NPS ARCHIVE
1965
SWANSON, H.

ANALYSIS OF INELASTIC ALPHA PARTICLE SCATTERING USING DIRECT INTERACTION THEORY

HERBERT FRANK SWANSON, JR.

DUDLEY KNOX LIBRARY
NAVAL POSTGRADUATE SCHOOL
MONTEREY CA 93943-5101

LIBRARY
U.S. NAVAL POSTGRADUATE SCHOOL
MONTEREY, CALIFORNIA

mon. 165

ANALYSIS OF INELASTIC ALPHA PARTICLE SCATTERING USING DIRECT INTERACTION THEORY

A Thesis

Submitted to the
Faculty of Miami University
in partial fulfillment of
the requirements for the degree of
Master of Arts

by

Herbert Frank Swanson Jr.
Miami University
Oxford, Ohio

1965

NPS Archive
1965
Swanson, H

~~S925~~

Library
U. S. Naval Postgraduate School
Monterey, California

DUDLEY KNOX LIBRARY
NAVAL POSTGRADUATE SCHOOL
MONTEREY CA 93943-5101

ACKNOWLEDGEMENTS

The Author is sincerely grateful to Dr. Joseph Priest for his encouragement and helpful discussions during the preparation of this text. The author wishes to express his indebtedness to Doctors J. Priest, and J. S. Vincent and R. W. Bercaw, NASA-Lewis Research Center for the use of their data prior to publication.

Appreciation is also expressed to Dr. George Arfken, chairman of the Department of Physics for unlimited use of the departmental facilities, and to the Miami University Computer Center for making available computer time.

ABSTRACT

ANALYSIS OF INELASTIC ALPHA PARTICLE SCATTERING

USING DIRECT INTERACTION THEORY

by

Herbert Frank Swanson Jr.

The theory of the direct interaction is discussed and its applicability to inelastic alpha particle scattering to low lying states is shown. Two results of this theory are developed; (1) the diffraction model of Blair and (2) the distorted wave Born approximation, where the phenomenological approximation to the optical wave function suggested by McCarthy and Pursey is used. Fits to $Ar^{36}(\alpha, \alpha')Ar^{36*}$ data are shown for 32.8 and 41.0 Mev(Lab) alpha particles. In the diffraction model analysis, R_0 = 6.45 Fermis for both energies. $\beta_2 R_0$ = .82 Fermis for the lower energy data and .71 Fermis for the higher energy data. $\beta_3 R_0$ = .67 and .56 Fermis for the lower and higher energy, respectively.

CONTENTS

LIST OF FIGURES

CHAPTER I

THE DIRECT INTERACTION

There are two basic mechanisms by which a nuclear reaction can proceed; (1) the direct interaction, and (2) the formation of a compound nucleus. The theoretical angular distribution predicted by each mechanism is different. To see the nature of these two processes, consider the incident particle to be a nucleon. Thus the independent-particle shell model is a good approximation to the nuclear structure. Here the many body problem is replaced by an effective single particle potential which describes the force acting on each particle, this potential being different for different shell model configurations.

As a particle enters the nucleus, it experiences a potential described by the other A nucleons in their ground state configuration. The projectile may then simply enter the nucleus, be deflected by the nuclear potential, and emerge again at a different angle but with the same energy in the center of mass system. This is called direct elastic scattering. If a reaction ensues, the energy transferred to the nucleus is determined by the interaction of the incoming nucleon with one in some shell configuration and by

the nuclear density. This later fact modifies the problem as neither of the colliding particles can have an energy below that of the ground state energy of the nucleus. This is because the hole left by the struck nucleon is the only level available below the ground state energy. The case where the incident particle replaces the struck nucleon is called exchange scattering and will not be considered here. Those nucleons excited to high level states should be described by a different potential than those excited to low lying states, hence the effective potential is a function of the bombarding energy. If either of the colliding nucleons has an energy greater than its separation energy, it may leave the nucleus without further reaction save deflection by the nuclear potential. This is described as a direct interaction and usually occurs in a time comparable with the time it takes the projectile to traverse the nucleus. If, on the other hand, neither of the nuclei has energy sufficient for separation, both will undergo further collisions eventially spreading their energy over the whole nucleus. For a time, the nuclear state will grow increasingly complex until it reaches a statistical equilibrium. This state will be a complicated admixture of shell-model and collective configurations involving excitation of the many degrees of freedom of the nucleus. Eventually

enough energy will be concentrated in a particle or group of particles to allow escape from the nucleus and upon emission the nucleus will decay into some new excited state. This process is referred to as scattering via the formation of a compound nucleus and is usually characterized by the long time necessary for its completion, which is about a thousand times that for the direct interaction. Since these two processes exhibit widely different cross sections, a method for separation into the direct interaction and compound nucleus components is desirable. No suitable technique, however, has been found. Instead reactions are chosen which are predominently either direct interaction or compound nucleus types and where errors made in neglecting the other are small.

Direct interaction processes involve only a few of the many degrees of freedom of the nucleus. The minimum number, in fact, would be just those necessary to specify the ingoing and outgoing channels of the reaction. The concept of the direct interaction can be extended to any projectile if it excites only one degree of freedom in the nucleus specifically a collective degree of freedom. For example, it would certainly be a direct interaction if the projectile bounced off a spheroidal nucleus, leaving it rotating. These rotational states are typically low lying states.

Compound nucleus reactions, however, involve many degrees of freedom due to the complicated nature of the nuclear excited state. It would then be fortunate to have a detector capable of indicating the number of degrees of freedom in the wave functions of the intercepted particles, but alas, none has ever been found. In a typical compound nucleus reaction the probability of a particle being emitted leaving the nucleus in a low lying state is greater than those leaving the nucleus in the higher excited states. This is due to the difficulty of barrier penetration which inhibits the emission of particles with energies just above the separation energy. Figure (1-1) shows a rough plot of the number of particles emitted vs. the excitation energy of the nucleus, where ϵ_G is the ground state energy.

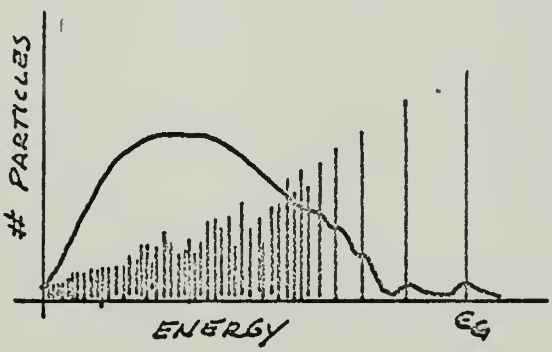


FIGURE (1-1)

The vertical lines show the discreet spectrum. The detector, however, does not possess perfect resolution

and the energy spectrum it sees is averaged over an interval ΔE. The solid curve shows such an averaging. Thus it is seen that although the compound nucleus would rather emit particles leaving the energy of the nucleus near its ground state, the majority of the flux occurs in the higher lying states.

Hence it is seen that inelastic scattering to low lying states should go predominantly by the direct interaction process. This theory will be developed in the following chapters.

CHAPTER II

THE SCATTERING CROSS SECTION

Introduction

A useful concept in relating theory to experiment is the differential cross section. In the usual picture of scattering, a collimated beam of particles impinges upon a target. The number of particles per second, dN, scattered into an element of solid angle is proportional to the incident flux N and the element of solid angle $d\Omega$.

$$dN = \sigma(\theta\phi)\, N\, d\Omega \tag{2-1}$$

The constant of proportionality $\sigma(\theta,\phi)$ has dimensions of area and is called the differential cross section. It can be a function of θ and ϕ the angular coordinates of $d\Omega$, but is usually only a function of θ taking the polar axis as the beam direction. The cross section is related to Schrodinger's equation in the following manner. At distances far from the scattering center, the wave function describing the scattered particles can be written as the sum of a plane wave corresponding to the incident beam and an out going spherical wave due to the

scattering.

$$\Psi = e^{i \underline{k}_F \cdot \underline{r}} + \chi_{SCAT}$$

$$\chi_{SCAT} = f(\theta\phi)\,\frac{e^{i k_F r}}{r} \tag{2-2}$$

The function $f(\theta\phi)$ is known as the scattering amplitude and contains all of the angular dependence of the scattered beam.

The quantum mechanical expression for the scattered flux is given by[1]

$$S_{SCAT} = -\frac{i\hbar}{2m_F}\left[\chi^*_{SCAT}\nabla\chi_{SCAT} - \chi_{SCAT}\nabla\chi^*_{SCAT}\right] \tag{2-3}$$

where m_F is the reduced mass of the scattered system. Applying this to (2-2), an expression is obtained giving the number of particles being scattered / unit area / sec.

$$S_{SCAT} = \frac{\hbar k_F}{m_F}\,\frac{|f(\theta\phi)|^2}{r^2} \tag{2-4}$$

where, in the asymptotic region considered, only the radial component contributes significantly to the flux.

The number of scattered particles passing through an element of area dA of a sphere of radius r located concentric with the scattering center is just dN.

$$dN = (S_{SCAT})\, dA = \frac{\hbar k_F}{m_F} \left| f(\theta\phi) \right|^2 d\Omega \qquad (2\text{-}5)$$

where $$d\Omega = \frac{dA}{r^2}$$

The incident flux is computed using an expression similar to (2-3), where the incident wave is $e^{ik_I z}$

$$S_{INC} = \frac{\hbar k_I}{m_I} \equiv N \qquad (2\text{-}6)$$

The quantity dN is then divided by the incident flux and the resulting expression compared with (2-1) to obtain the scattering cross section;

$$\sigma(\theta\phi) = \left(\frac{m_I}{m_F}\right)\left(\frac{k_F}{k_I}\right)\left| f(\theta\phi) \right|^2. \qquad (2\text{-}7)$$

In the above intuitive argument, it should be noted that any interference between the incident and scattered waves has been ignored. It would thus appear that (2-7) is inaccurate and of limited usefulness. The incident beam however, has a macroscopically small cross sectional area and thus would only interfere with the scattered wave at both very small and very large scattering angles. Since the density of particles in these beams is small, they will not greatly influence

one another even in these cases. For a more rigorous treatment, the reader is directed to reference [2], where it will be noted that the result is the same.

It is the purpose of the remaining sections of this chapter to obtain expressions for the scattering amplitude, using the Schrodinger equation with an appropriate potential. As this is a study of the direct interaction, the Hamiltonian will exclude all flux that would go into populating the states of the compound nucleus and would itself describe only those few internal degrees of freedom pertinent to the direct interaction. Typically, it would describe the motions of the one or two nucleons to which momentum had been transferred.

The Schrodinger Equation

Consider the Shrodinger equation in the center of mass system. The subscript f means "with respect to the final channel f ."

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + U(r_f) + V(r_f, \xi_f) + \mathcal{H}(\xi_f)\right]\Psi = E\,\Psi \qquad (2\text{-}8)$$

The first two terms on the left are the kinetic energy and the optical interaction potential of the separating fragments, and $\mathcal{H}(\xi_f)$ is the Hamiltonian describing their internal motions. $V(r_f, \xi_f)$ is the residual

interaction energy, with respect to the final state channel, which gives rise to transitions to excited states of the residual nucleus. The optical potential is of the form

$$\frac{zZe^2}{r} + \frac{U - iV}{\left(1 + e^{(R-r)/d}\right)}$$

where the term on the left is the Coulomb potential for a point charge Z and an incident particle of charge z. The term on the right is the phenomenological complex nuclear potential first introduced by Saxon and thus bears his name. While it will suffice here to give a plausibility argument for the complex nature of this potential, the reader is referred to [3] and [4] for a theoretical justification. The study of dispersion theory in physical optics has shown that absorption of light in a cloudy or semi-transparent material can be represented mathematically by the imaginary term of a complex index of refraction. In the study of nuclear reactions, the phase of a particle wave impingent on a potential is shifted as it passes through the region of the potential and it is known that the scattering amplitude is completely describable in terms of these phase shifts.[5] Completely analogous to the optical case, if these phase shifts are complex, the imaginary part would produce absorption of part of

the incident beam. It is then the imaginary part of the complex potential which gives rise to absorption. This term literally absorbs out all flux which is not described by the potential $V(r,\xi)$, such as the compound nucleus flux, thus eliminating it from the scattered beam. The denominator gives the radial dependence of the potential combining its short range nature with a diffuse edge. It will be mentioned in passing that many variations on this appear in the literature. However, they often contain so many parameters that the question arises as to whether or not the resultant fits contain any physics or are just attempts at curve fitting. The variables ξ_F (or ξ_I in the incident channel) are those few internal degrees of freedom considered. The Hamiltonian $\mathcal{H}(\xi_F)$ is assumed to have eigen functions $V_F(\xi_F)$ which are solutions of the equation

$$\mathcal{H}(\xi_F)\, V_F(\xi_F) = h\, V_F(\xi_F) \qquad (2\text{-}9)$$

To solve equation (2-8), it is first written with the two potential terms isolated.

$$\left(-\frac{\hbar^2}{2m_F}\nabla^2 + \mathcal{H}(\xi_F) - E\right)\Psi = -(U+V)\Psi \qquad (2\text{-}10)$$

To remove the ξ_F dependence, the equation is multiplied by the complex conjugate of $V_F(\xi_F)$ and integrated over the internal variables.

$$\left[-\frac{\hbar^2}{2m_F}\nabla^2 - E\right]\int V_F^* \Psi \, d\xi_F + \int V_F^* \mathcal{H}(\xi_F)\, d\xi_F$$

$$= -\int V_F^* (U+V)\, \Psi \, d\xi_F \tag{2-11}$$

Using (2-9) and the Hermitean nature of $\mathcal{H}(\xi_F)$, equation (2-11) can be written, using Dirac's "bra-ket" notation for the integrals as

$$(\nabla^2 + k^2)\psi_F = \frac{2m_F}{\hbar^2}\langle V_F | U+V | \Psi \rangle \tag{2-12}$$

$$\psi_F = \langle V_F(\xi_F) | \Psi \rangle \tag{2-13}$$

$$k_F^2 = \frac{2m_F}{\hbar^2}(E-h) \tag{2-14}$$

ψ_F is the coefficient in an expansion of Ψ into a set of eigen functions $V_F(\xi_F)$ and can be thought of as a projection of Ψ on the basis vector for the final channel " f ". The homogeneous part of Equation (2-12) is the Helmholtz wave equation with solutions[6]

$$\phi = e^{\pm i \underset{\sim}{k} \cdot \underset{\sim}{r}}$$

General Integral Equation Solution

Equation (2-12) is transformed into an integral equation through the use of a Green's function. It can be shown that

$$\psi_F = -\frac{2m_F}{\hbar^2}\int K(\underset{\sim}{r}_F,\underset{\sim}{r}_F')\langle v_F|U+V|\Psi\rangle d^3r' \qquad (2\text{-}15)$$

is a solution to (2-12), where the Green's function $K(\underset{\sim}{r}_F,\underset{\sim}{r}_F')$, is a solution to the equation

$$(\nabla^2 + k_F^2)\,K(\underset{\sim}{r}_F,\underset{\sim}{r}_F') = -\,\delta(\underset{\sim}{r}_F - \underset{\sim}{r}_F') \qquad (2\text{-}16)$$

as long as $\underset{\sim}{r}_f \neq \underset{\sim}{r}_f'$, (2-16) is just the wave equation mentioned earlier with outgoing solution

$$\phi = e^{i\underset{\sim}{k}_F\cdot\underset{\sim}{r}_F}$$

This is expanded in a series of Legendre polynomials[7]

$$e^{i\underset{\sim}{k}_F\cdot\underset{\sim}{r}_F} = \sum_{l=0}^{\infty} i^l(2l+1)j_l(k_F r_F)P_L(\cos\Theta) \qquad (2\text{-}17)$$

Using the Legendre addition theorem[8], (2-17) can be expressed in the space fixed coordinate system.

$$e^{i\underset{\sim}{k}_F\cdot\underset{\sim}{r}_F} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l} i^l\, j_l(k_F r_F)\, Y_{lm}^{*}(\theta'\phi')\, Y_{lm}(\theta\phi) \tag{2-18}$$

The primed and unprimed variables are the angular coordinates of $\underset{\sim}{k}_f$ and $\underset{\sim}{r}_f$ respectively. At large distances from the origin, the asymptotic form of $j_l(kr)$ may be used.

$$j_l(k_F r_F) \rightarrow \frac{1}{k_F r_F}\sin\left(k_F r_F - \frac{l\pi}{2}\right) \tag{2-19}$$

$$e^{i\underset{\sim}{k}_F\cdot\underset{\sim}{r}_F} \rightarrow \frac{4\pi}{k_F r_F}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} i^l \sin\left(k_F r_F - \frac{l\pi}{2}\right) Y_{lm}^{*}(\theta'\phi')\, Y_{lm}(\theta,\phi) \tag{2-20}$$

We proceed by developing a series solution for the Green's function. All "f" subscripts will be dropped until they again become necessary. The three dimensional Dirac delta function expressed in spherical polar coordinates is[9]

$$\delta(\underset{\sim}{r}-\underset{\sim}{r}') = -\frac{1}{r^2}\,\delta(r-r')\,\delta(\cos\theta-\cos\theta')\,\delta(\phi-\phi') \tag{2-21}$$

The spherical harmonics form a complete set over the surface of a sphere and a mathematical statement of this is the following "closure relation"[10]

$$\delta(\cos\theta-\cos\theta')\,\delta(\phi-\phi') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y^{*}_{lm}(\theta'\phi')\,Y_{lm}(\theta,\phi) \qquad (2\text{-}22)$$

The total delta function can then be expressed in terms of this series. The closure relation suggests the following expansion for the Green's function.

$$K(\underset{\sim}{r},\underset{\sim}{r}') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A^{*}(\theta'\phi')\,g_l(r,r')\,Y_{lm}(\theta\phi) \qquad (2\text{-}23)$$

This is now inserted into the defining equation, (2-16), where the Laplacian in spherical coordinates is

$$\nabla^2\Psi = \frac{1}{r^2}\frac{\partial^2(r\Psi)}{\partial r^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Psi}{\partial\phi^2} \qquad (2\text{-}24)$$

Equation (2-16) becomes

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l} A^{*}(\theta\phi)\left\{\left[\frac{1}{r}\frac{d^2}{dr^2}\left(r g_l(r,r')\right) + k^2 g_l(r,r')\right]\right.$$

$$\left. + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y_{lm}}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y_{lm}}{\partial\phi^2}\right]\right. \qquad (2\text{-}25)$$

$$= -\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{r^2}Y^{*}_{lm}(\theta',\phi')\,Y_{lm}(\theta\phi)$$

The spherical harmonics are a solution to the following differential equation.

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y_{\ell m}}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 Y_{\ell m}}{\partial\phi^2}+\ell(\ell+1)Y_{\ell m}=0$$

This is solved for the second set of square brackets and the two series are equated term by term.

$$A^*(\theta'\phi') = Y^*_{\ell m}(\theta'\phi') \tag{2-26}$$

$$\frac{d^2}{dr^2}\left(r g_\ell(r,r')\right) + \left(k^2 - \frac{\ell(\ell+1)}{r^2}\right) r g_\ell(r,r') = -\frac{1}{r}\delta(r-r') \tag{2-27}$$

Let

$$G_\ell(r,r') \equiv r g_\ell(r,r') \tag{2-28}$$

The radial Green's function is a solution to

$$\frac{d^2 G_\ell(r,r')}{dr^2} + \left(k^2 - \frac{\ell(\ell+1)}{r^2}\right) G_\ell(r,r') = -\frac{1}{r}\delta(r-r') \tag{2-29}$$

Consider the homogeneous part of this equation.

$$\frac{d^2}{dr^2} f_\ell(kr) + \left(k^2 - \frac{\ell(\ell+1)}{r^2}\right) f_\ell(kr) = 0 \tag{2-30}$$

As this is a second order differential equation, it has two independent solutions. The radial Green's function, $G_\ell(r,r')$, is built from these two solutions by the following prescription.

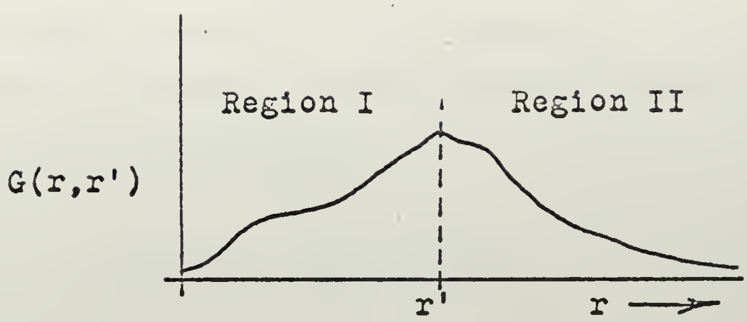


FIGURE (2-1)

Figure (2-1) shows a plot of some general $G(r,r')$ as a function of r. It must be a solution to (2-30) in both regions, satisfying the boundary condition at the origin in region I and the boundary condition at infinity in region II. While it is required that the magnitude of $G_\ell(r,r')$ be continuous across the boundary at $r = r'$, there will be a discontinuity in the slope. The magnitude of this discontinuity can be obtained by integrating equation (2-29) from $r'-\epsilon$ to $r'+\epsilon$ and taking the limit as $\epsilon \to 0$.

$$\lim_{\epsilon \to 0} \int_{r'-\epsilon}^{r'+\epsilon} d\left(\frac{d}{dr} G_\ell(r,r')\right) + \int_{r'-\epsilon}^{r'+\epsilon} \left(k^2 - \frac{\ell(\ell+1)}{r^2}\right) G_\ell(r,r')\,dr = -\int_{r'-\epsilon}^{r'+\epsilon} \frac{1}{r}\,\delta(r-r')\,dr$$

$$\left.\frac{d}{dr} G_\ell(r,r')\right|_{r'+} - \left.\frac{d}{dr} G_\ell(r,r')\right|_{r'-} = -\frac{1}{r'} \qquad (2\text{-}31)$$

The second integral contributes nothing, as $G_\ell(r,r')$ is continuous across the discontinuity at r'. Two solutions to equation (2-30) will be obtained. Introduce the substitution

$$f_\ell(kr) = \sqrt{kr}\, Z(kr)$$

This is inserted into the homogeneous equation (2-30), yielding

$$\frac{d^2Z}{dr^2} + \frac{1}{r}\frac{dZ}{dr} + \left(k^2 - \frac{(\ell+\frac{1}{2})^2}{r^2}\right)Z = 0 \qquad (2\text{-}32)$$

This is Bessel's equation of index $\ell+\frac{1}{2}$ and

$$Z(kr) = J_{\ell+\frac{1}{2}}(kr) \, .$$

Multiply through by the constant $\sqrt{\frac{\pi}{2}}$

$$f_\ell(kr) = kr\sqrt{\frac{\pi}{2kr}}\, J_{\ell+\frac{1}{2}}(kr) = kr\, j_\ell(kr) \qquad (2\text{-}33)$$

where $j_\ell(kr)$ is the spherical Bessel function. Consideration of equation (2-17) shows this to be kr times the radial wave function. Here, $j_\ell(kr)$ is regular at the origin and thus satisfies the boundary condition in region I. The following relation exists between the spherical Bessel and Hankel functions.[11]

$$j_\ell(kr) = \frac{1}{2}\left(h_\ell^{(1)}(kr) + h_\ell^{(2)}(kr)\right) \qquad (2\text{-}34)$$

The functions $h_\ell^{(1)}$ and $h_\ell^{(2)}$ are, respectively, the spherical Hankel functions of type one and type two. They are both solutions to equation (2-30). Asymptotically $h_\ell^{(1)}$ goes as an outgoing spherical wave and $h_\ell^{(2)}$ as an incoming one. As r approaches infinity, there can be only outgoing scattered waves. Hence a solution satisfying the boundary condition in region II is $h_\ell^{(1)}(kr)$. The Green's function is written

$$G_\ell(r,r') = \begin{cases} C_1\, kr\, j_\ell(kr) & r<r' \\ C_2\, kr\, h_\ell^{(1)}(kr) & r>r' \end{cases} \tag{2-35}$$

subject to the condition imposed by (2-31).

$$C_2 \frac{d}{dr}\Big[kr\, h_\ell^{(1)}(kr)\Big]_{r=r'} - C_1 \frac{d}{dr}\Big[kr\, j_\ell(kr)\Big]_{r=r'} = -\frac{1}{r'} \tag{2-31a}$$

The Wronskian relation for these two functions can be used to evaluate the constants C_1 and C_2. Sturm-Liouville theory[12] gives the Wronskian of two independent solutions to equation (2-30) as

$$W\left(j_\ell(kr), h_\ell^{(1)}(kr)\right) = A \tag{2-36}$$

where A is a constant.

$$W(j_\ell, h_\ell^{(1)}) = (kr) j_\ell \frac{d}{dr}(kr\, h_\ell^{(1)})$$

$$- kr\, h_\ell^{(1)} \frac{d}{dr}(kr\, j_\ell) \qquad (2\text{-}37)$$

As this is true for all r, the asymptotic forms of these expressions may be used in computation. The asymptotic form of $j_\ell(kr)$ was already given in (2-19) and for large r

$$h_\ell^{(1)}(kr) \rightarrow \frac{1}{kr} e^{i(kr - \frac{\ell\pi}{2})} \qquad (2\text{-}38)$$

Calculation of the Wronskian yields

$$A = k$$

A comparison of (2-31a) with (2-37) shows

$$C_1 = h_\ell^{(1)}(kr')$$

$$C_2 = j_\ell(kr')$$

The Green's function becomes

$$G_\ell(r, r') = kr\, j_\ell(kr_<)\, h_\ell^{(1)}(kr_>) \qquad (2\text{-}39)$$

$r_<$ is the smaller of r and r′

$r_>$ is the greater of r and r′

Combining equations (2-23), (2-26), (2-28), and (2-39), the total Green's function is obtained.

$$K(\underset{\sim}{r},\underset{\sim}{r}') = k\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} j_\ell(kr_<)\, h_\ell^{(1)}(kr_>) \times Y_{\ell m}^{*}(\theta'\phi')\, Y_{\ell m}(\theta\phi) \tag{2-23a}$$

Equation (2-23a) is then used in (2-15) to give the solution for ψ_F.

The General Scattering Amplitude

As was discussed in the introduction of this chapter, the scattering amplitude is the coefficient of the outgoing scattered spherical wave. Equation (2-15) will be used to find the asymptotic form of ψ_F. Equation (2-23a) can be simplified for large r, r being $\gg$ r$'$, where $h_\ell^{(1)}$ can be replaced by its asymptotic form given in (2-39). The Green's function becomes

$$K(\underset{\sim}{r}_F,\underset{\sim}{r}_F') = \frac{e^{ik_F r_F}}{r_F}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(-i)^\ell j_\ell(k_F r_F')\, Y_{\ell m}^{*}(\theta'\phi')\, Y_{\ell m}(\theta\phi) \tag{2-23b}$$

where the primed angular coordinates refer to $\underset{\sim}{r}_F'$ and the unprimed ones to $\underset{\sim}{r}_F$ which is taken to be along the vector $\underset{\sim}{k}_F$. Consider the expression in (2-18). Taking the complex conjugate yields

$$e^{-i\underset{\sim}{k}_F\cdot\underset{\sim}{r}_F} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(-i)^l j_l(k_F r_F)\,Y^*_{lm}(\theta'\phi')\,Y_{lm}(\theta\phi)$$

where the primed coordinates refer to $\underset{\sim}{k}_F'$ and the unprimed ones to $\underset{\sim}{r}_F$.

$$K(r_F,\underset{\sim}{r}_F') = \frac{1}{4\pi r_F}e^{ik_F r_F}e^{-i\underset{\sim}{k}_F\cdot\underset{\sim}{r}_F'}$$

Using equation (2-15)

$$\psi_F \to e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} - \frac{1}{r_F}e^{ik_F r_F}\frac{m_F}{4\pi\hbar^2}\int V_F^* e^{-i\underset{\sim}{k}_F\cdot\underset{\sim}{r}_F}(U+V)\Psi\, d^3r\, d^3\underset{\sim}{\xi}$$

The plane wave $e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}}$ is a solution to the homogeneous part of (2-12) and can always be added to the scattered solution. The scattering amplitude is then equated with the coefficient of $\frac{1}{r_F}e^{ik_F r_F}$

$$f(\theta\phi) = -\frac{m_F}{4\pi\hbar^2}\left\langle V_F e^{i\underset{\sim}{k}_F\cdot\underset{\sim}{r}_F}\middle| U+V \middle| \Psi\right\rangle \qquad (2\text{-}40)$$

where the Dirac notation is again used to indicate integration over all internal and external variables.

The Elastic Scattering Amplitude

In a description of only the elastic component of the scattering, the potential $V(r_F\,\underset{\sim}{\xi})$ which describes transitions to excited states is set equal

to zero and the interpretation of the imaginary part of the complex potential is broadened to include absorption of the entire inelastic flux. The initial and final channels are then the same and the outgoing wave function Ψ is

$$\Psi = V_I(\xi_I)\,\chi_I^{(+)}(\underset{\sim}{k}_I, \underset{\sim}{r}_I) \tag{2-41}$$

where $V_I(\xi_I)$ and $\chi_I^{(+)}(\underset{\sim}{k}_I, \underset{\sim}{r}_I)$ are, respectively, the internal and elastic scattering wave functions for the initial channel. $\chi_I^{(+)}$ satisfies the equation

$$\left(-\frac{\hbar^2}{2m_I}\nabla^2 + U\right)\chi_I^{(+)}(k_I, r_I) = E\chi_I^{(+)}(k_I, r_I)$$

The elastic scattering amplitude is obtained from (2-40)

$$f_{EL}(\theta\phi) = -\frac{m_F}{2\pi\hbar^2}\left\langle V_F e^{i\underset{\sim}{k}_F\cdot\underset{\sim}{r}_F} \middle| U \middle| V_I\chi_I^{(+)}\right\rangle \tag{2-42}$$

Inelastic Scattering - Distorted Waves

The direct interaction inelastic scattering component is to be isolated from the scattered wave and using techniques developed in the first part of this chapter, the inelastic scattering amplitude will be found. Equation (2-8) is re-written with the potential which describes the inelastic transition

isolated.

$$\left(-\frac{\hbar^2}{2m_F}\nabla^2 + U + \mathcal{H}(\xi_F) - E\right)\Psi = -V(r_F,\xi_F)\Psi$$

Using (2-9) and the Hermitean nature of $\mathcal{H}(\xi_F)$, the ξ_F dependence is removed in the same manner as for (2-12)

$$\left(\nabla^2 - \mathcal{U} + k_F^2\right)\psi_F = \frac{2m_F}{\hbar^2}\langle V_F | V(r_F,\xi_F) | \Psi\rangle \quad (2\text{-}43)$$

$$\mathcal{U} = \frac{2m_F}{\hbar^2}U(r_F) \quad (2\text{-}44)$$

k_F^2 and ψ_F are discussed after equation (2-12) and will be reiterated here for the readers' convenience.

$$\psi_F \equiv \langle V_F(\xi_F) | \Psi\rangle$$

$$k_F^2 = \frac{2m_F}{\hbar^2}(E - h)$$

The homogeneous part of (2-43) has as one of its eigen solutions, the elastic scattering wave function discussed in relation to equation (2-41). A series solution corresponding to an outgoing wave will be obtained. Consider the homogeneous equation with the f subscript dropped as before.

$$\left(\nabla^2 - \mathcal{U} + k_F^2\right)\chi^{(+)} = 0 \quad (2\text{-}45)$$

The form of the desired solution is such as to reduce to expression (2-18) when the optical potential is removed, as certainly (2-45) reduces to the Helmholtz wave equation. As the optical potential is a function of only the radial variable, $\chi^{(+)}(\underset{\sim}{k}, \underset{\sim}{r})$ will have azimuthal symmetry if $\underset{\sim}{k}$ is assumed to be along the z-axis. (Note expression (2-17)). The discussion following Equation (2-8) gave the optical potential to be of the form

$$V_{COULOMB} + V_{NUCLEAR}$$

A plausibility argument for the complex form of the nuclear potential was given as well as an expression, typical of those appearing in the literature. No explicit form of V_N is necessary in the following calculations. However, it will be assumed that the nuclear forces are short ranged and thus V_N will rapidly approach zero for distances greater than the nuclear radius. Equation (2-45) becomes

$$\left[\nabla^2 + k^2 - \frac{2\alpha k}{r} - \frac{2m}{\hbar^2} U_N\right]\chi^{(+)} = 0 \quad (2\text{-}45a)$$

where $$\alpha = \frac{m z Z e^2}{\hbar^2 k} = \frac{z Z e^2}{\hbar v} \qquad (2\text{-}46)$$

Coulomb Wave Functions

Consider the special case of zero nuclear potential. This situation would be encountered if the impingent beam lacked sufficient energy to overcome the coulomb barrier and penetrate into the nuclear material. The eigen solutions become coulomb wave functions, satisfying the equation

$$\left[\nabla^2 + k^2 - \frac{2\alpha k}{r}\right] \phi_c(\underset{\sim}{r}) = 0 \qquad (2\text{-}47)$$

The second chapter of reference [3] is an excellent source for information regarding solutions of this equation. It turns out that (2-47) has a regular solution of the form

$$\phi_c = e^{ikz} f(r-z) \qquad (2\text{-}48)$$

If z is the polar axis, then $z = r \cos\theta$ and $f(r-z)$ has azimuthal symmetry. Using (2-24), the Laplacian is expanded in spherical polar coordinates and the following substitution is made.

$$\phi_c = e^{ikr\cos\theta} f(u)$$

$$u = r(1-\cos\theta)$$

The substitution is carried out using the chain rule.

$$\frac{\partial f(u)}{\partial x} = \frac{df(u)}{du}\frac{\partial u}{\partial x} \tag{2-49}$$

The results in the following differential equation for $f(u)$

$$\left[u\frac{d^2}{du^2} + (1-iku)\frac{d}{du} - \alpha k\right]f(u) = 0 \tag{2-50}$$

The factor k is put completely within the variable with the transformation

$$\nu = iku \tag{2-51}$$

Equation (2-50) then becomes

$$\left[\nu\frac{d^2}{d\nu^2} + (1-\nu)\frac{d}{d\nu} + i\alpha\right]f(\nu) = 0 \tag{2-52}$$

Equation (2-52) is in the form of the confluent hypergeometric equation which is solved in appendix A. A comparison of equation (A-1) and its solution (A-4) with (2-52) gives

$$f(\nu) = {}_1F_1(-i\alpha|1|\nu)$$

and

$$\phi_c = C e^{ikz} {}_1F_1(-i\alpha|1|ik(r-z)) \qquad (2\text{-}53)$$

A consideration of equation (A-4) shows it to be non-singular at the origin and therefore the solution ϕ_c is the regular coulomb wave function. The constant C is determined so that asymptotically

$$\phi_c = e^{ikz} + \phi_{SCAT}$$

where as usual ϕ_{scat} is the scattered outgoing spherical wave. In appendix A, the following asymptotic solution (A-19) is derived.

$${}_1F_1(a|b|z) \sim \frac{\Gamma(b)}{\Gamma(b-a)}(-z)^{-a} + \frac{\Gamma(b)}{\Gamma(a)} e^{z} z^{a-b}$$

making the substitutions, equation (2-53) becomes

$$\phi_c \to \frac{C e^{\pi\alpha/2}}{\Gamma(1+i\alpha)} e^{i(kz-\alpha \ln(k(r-z)))} - \frac{C\alpha e^{\alpha\pi/2} e^{i(kr-\alpha \ln(k(r-z)))}}{k(r-z)\,\Gamma(1-i\alpha)} \qquad (2\text{-}54)$$

the logarithm is a slowly varying function with respect to its argument. Hence at large distances from the scatterer, the first of these two terms is essentially

an outgoing plane wave and the second an outgoing spherical wave. The coefficient of the plane wave term was defined as being unity and thus the value of C is determined.

$$C = e^{-\pi\alpha/2}\Gamma(1+i\alpha) \tag{2-55}$$

Equation (2-47) is also separable in spherical coordinates, thus leading to a decomposition of ϕ_c into partial waves.

Let $$\phi_c = \frac{F(r)}{r}\,\Theta(\theta)$$

When this is substituted in (2-53) two equations for F and Θ are obtained.

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \ell(\ell+1)\Theta = 0 \tag{2-56}$$

$$\frac{d^2F_\ell}{dr^2} + \left(k^2 - \frac{2\alpha k}{r} - \frac{\ell(\ell+1)}{r^2}\right)F_\ell = 0 \tag{2-57}$$

where $\ell(\ell+1)$ is the separation constant. Equation (2-56) is Legendre's equation which has Legendre polynomials as its solutions.

$$\Theta = P_\ell(\cos\theta)$$

In equation (2-51) let

$$\rho = -2ikr \tag{2-58}$$

With this transformation, (2-57) becomes

$$\frac{d^2F_\ell}{d\rho^2} - \left(\frac{1}{4} + \frac{i\alpha}{\rho} + \frac{\ell(\ell+1)}{\rho^2}\right)F_\ell = 0 \qquad (2\text{-}59)$$

Let
$$F_\ell = e^{-\rho/2}(i\rho/2)^{\ell+1}V_\ell(\rho) \qquad (2\text{-}60)$$

After some manipulation, the equation for V_ℓ becomes

$$\rho\frac{d^2V_\ell}{d\rho^2} + (2\ell+2-\rho)\frac{dV_\ell}{d\rho} - (\ell+1+i\alpha)V_\ell = 0 \qquad (2\text{-}61)$$

A comparison of (2-61) to (A-1) shows V_ℓ to be a solution of the confluent hypergeometric equation where,

$$V_\ell = {}_1F_1(\ell+1+i\alpha\,|\,2\ell+2\,|\,\rho) \qquad (2\text{-}62)$$

The radial coulomb wave function becomes

$$F_\ell(kr) = A_\ell e^{ikr}(kr)^{\ell+1}{}_1F_1(\ell+1+i\alpha\,|\,2\ell+2\,|\,-2ikr) \qquad (2\text{-}63)$$

where the A_ℓ are determined by the same asymptotic considerations as before, namely that ϕ_c' behaves as an outgoing plane wave at large r. The asymptotic expression for F_ℓ is calculated in the same manner as for equation (2-53). By substitution of the proper parameters into (A-19), the following result is obtained.

$$F_\ell \approx \frac{A_\ell e^{\alpha\pi/2}\Gamma(2\ell+2)}{2^\ell (2i)}\left\{ \frac{e^{i(kr-\alpha\ln 2kr-\frac{\ell\pi}{2})}}{\Gamma(\ell+1-i\alpha)} - \frac{e^{-i(kr-\alpha\ln 2kr-\frac{\ell\pi}{2})}}{\Gamma(\ell+1+i\alpha)}\right\} \qquad (2\text{-}64)$$

The general theory of complex variables gives

$$\Gamma(\ell+1\pm i\alpha) = |\Gamma(\ell+1+i\alpha)|\, e^{\pm i\sigma_\ell}$$

where $\sigma_\ell = \arg(\Gamma(\ell+1+i\alpha))$ (2-65)

with this substitution, expression (2-64) simplifies to

$$F_\ell \approx \frac{A_\ell e^{\alpha\pi/2}\Gamma(2\ell+2)}{2^\ell|\Gamma(\ell+1+i\alpha)|}\sin\left(kr-\alpha\ln 2kr-\frac{\ell\pi}{2}+\sigma_\ell\right) \qquad (2\text{-}66)$$

A consideration of the plane wave expansion in (2-17) shows the corresponding radial wave function to be $j_\ell(kr)$, with its asymptotic form given in (2-19). The constants A_ℓ are chosen so that these two functions approach each other as $r \to \infty$

$$A_\ell = \frac{2^\ell|\Gamma(\ell+1+i\alpha)|\, e^{-\alpha\pi/2}}{\Gamma(2\ell+2)} \qquad (2\text{-}67)$$

The total solution for ϕ_c is then the sum of all the partial waves. This is shown by using expressions (2-53) and (2-55) for ϕ_c and the contour integral

representation of the confluent hypergeometric function given in (A-7)

$$\phi_c = \frac{e^{-\pi\alpha/2} e^{ikz}}{(1-e^{-2\pi\alpha})\Gamma(-i\alpha)} \int e^{ik(r-z)t} t^{-i\alpha-1}(1-t)^{i\alpha} dt$$

$$= \frac{e^{-\pi\alpha/2}}{(1-e^{-2\pi\alpha})\Gamma(-i\alpha)} \int e^{ikrt} e^{ikz(1-t)} t^{-i\alpha-1}(1-t)^{i\alpha} dt \quad (2\text{-}68)$$

using equation (2-17), the exponential $e^{ikz(1-t)}$ is expanded into a series of Legendre polynomials where $z = r \cos\theta$

$$\phi_c = \sum_{\ell=0}^{\infty} i^\ell (2\ell+1)\, \xi_\ell(kr)\, P_\ell(\cos\theta) \quad (2\text{-}69)$$

Where

$$\xi_\ell(kr) = \frac{e^{-\alpha\pi/2}}{(1-e^{-2\pi\alpha})\Gamma(-i\alpha)} \int e^{ikrt} j_\ell(kr(1-t))\, t^{-i\alpha-1}(1-t)^{i\alpha} dt \quad (2\text{-}70)$$

a consideration of equations (2-30), (2-33), (2-57), (2-63), and (2-67) with equal to zero in the last three, gives

$$x\, j_\ell(x) = \frac{2^\ell \Gamma(\ell+1)}{\Gamma(2\ell+1)} e^{ix} x^{\ell+1} {}_1F_1(\ell+1 \,|\, 2\ell+2 \,|\, -2ix) \quad (2\text{-}71)$$

using the series form of the confluent hypergeometric function (A-4), equation (2-71) becomes

$$j_\ell(x) = 2^\ell e^{ix} x^\ell \sum_{s=0}^{\infty} \frac{\Gamma(\ell+1+s)(-2ix)^s}{\Gamma(2\ell+2+s)\, s!} \qquad (2\text{-}72)$$

Letting $x = kr(1-t)$, (2-72) is substituted into (2-70) and the order of summation and integration is reversed. This gives

$$\xi_\ell(kr) = \frac{2^\ell e^{-\pi\alpha/2}(kr)^\ell e^{ikr}}{(1-e^{-2\pi\alpha})\Gamma(-i\alpha)}$$

$$\times \sum_{s=0}^{\infty} \frac{\Gamma(\ell+1+s)}{\Gamma(2\ell+2+s)} \left\{ \int t^{-i\alpha-1}(1-t)^{i\alpha+\ell+s}\,dt \right\} \frac{(-2ikr)^s}{s!} \qquad (2\text{-}73)$$

The integral given in the parenthesis is a contour integral representation of the beta function as given in expression (A-6)

$$\int t^{-i\alpha-1}(1-t)^{i\alpha+\ell+s}\,dt = (1-e^{-2\pi\alpha}) \frac{\Gamma(-i\alpha)\Gamma(\ell+s+1+i\alpha)}{\Gamma(\ell+s+1)}$$

with this substitution, (2-73) becomes

$$\xi_\ell(kr) = 2^\ell e^{-\pi\alpha/2}(kr)^\ell e^{ikr} \frac{\Gamma(\ell+1+i\alpha+s)}{\Gamma(2\ell+2)}$$

$$\times \sum_{s=0} \frac{\Gamma(\ell+1+i\alpha+s)\,\Gamma(2\ell+2)}{\Gamma(2\ell+2+s)\,\Gamma(\ell+1+i\alpha)} \frac{(-2ikr)^s}{s!}$$

By equation (A-4), this is written

$$\xi_\ell(kr) = 2^\ell e^{-\pi\alpha/2}(kr)^\ell e^{ikr}$$

$$\times \frac{\Gamma(\ell+1+i\alpha)}{\Gamma(2\ell+2)} {}_1F_1(\ell+1+i\alpha|2\ell+2|-2ikr) \tag{2-74}$$

(2-74) is compared with (2-63) and (2-67), giving

$$\xi_\ell(kr) = \frac{F_\ell(kr)\,e^{i\sigma_\ell}}{kr} \tag{2-75}$$

where σ_ℓ is defined in expression (2-65). Expressions (2-69) and (2-75) are combined and the partial wave expansion of ϕ_c is obtained

$$\phi_c = \frac{1}{kr}\sum_{\ell=0}^{\infty} i^\ell(2\ell+1)e^{i\sigma_\ell}F_\ell(kr)P_\ell(\cos\theta) \tag{2-76}$$

Optical Wave Functions

The eigen solution to equation (2-45a) corresponding to an outgoing wave will now be obtained. It was shown in the previous section on coulomb waves, that if the nuclear potential is neglected, the equation is separable in spherical polar coordinates, and since V_N is a function of only the r variable, the separability is unaffected. If the Laplacian is expanded and the variables separated, the angular wave function is unchanged. The equation for the radial coordinate is

$$\left(\frac{d^2}{dr^2}+k^2-\frac{2mV_N}{\hbar^2}-\frac{2\alpha k}{r}-\frac{\ell(\ell+1)}{r^2}\right)\mathcal{F}_\ell(kr)=0 \quad (2\text{-}77)$$

The total solution is

$$\chi^{(+)}(kr)=\frac{1}{r}\sum_{\ell=0}^{\infty}C_\ell\,\mathcal{F}_\ell(kr)\,P_\ell(\cos\gamma) \qquad (2\text{-}78)$$

where, as usual, the constants C_ℓ are determined from the asymptotic considerations.

It was shown in equation (2-64) that the regular asymptotic form of the radial coulomb wave function could be built from a linear combination of the two asymptotic exponential solutions

$$e^{\pm i(kr-\alpha\ln 2kr-\frac{\ell\pi}{2}+\sigma_\ell)}$$

where the plus and minus signs correspond to outgoing and ingoing waves, respectively. As the nuclear potential was postulated as being short ranged, these may also be used in determining the asymptotic form of $\mathcal{F}_\ell(kr)$. This is seen by substituting the exponentials into equation (2-77) with the following result.

$$\left(\frac{2mV_N}{\hbar^2} + O\left|\frac{1}{r^2}\right|\right) e^{\pm i\left(kr - \alpha \ln 2kr - \frac{\ell\pi}{2} + \sigma_\ell\right)} \rightarrow 0$$

It is seen that as long as V_N approaches zero at least as fast as r^{-2}, an asymptotic form of $\mathcal{F}_\ell(kr)$ can also be formed from these exponentials with error certainly no greater than that inherent in the coulomb solutions. The phenomenological nuclear potential postulated in the earlier part of this chapter meets this requirement. From consideration of the corresponding coulomb solution, the regular asymptotic solution to the radial optical wave function is

$$\mathcal{F}_\ell(kr) \rightarrow \sin\left(kr - \alpha \ln 2kr - \frac{\ell\pi}{2} + \sigma_\ell + \delta_\ell\right) \qquad (2\text{-}79)$$

where δ_ℓ is an additional phase shift which the wave would experience in passing through the region of the nuclear potential. The constants C_ℓ are now determined from the following two considerations: (1) The nuclear potential, being short ranged cannot affect the incoming asymptotic solution and, (2) When V_N is set equal to zero $\chi^{(+)}$ must reduce to the coulomb wave. As it was shown that the solution could be written as a linear combination of the ingoing and outgoing coulomb waves,

the first requirement becomes

$$A_\ell \sin(kr - \alpha \ln 2kr - \frac{\ell\pi}{2} + \sigma_\ell + \delta_\ell) =$$

$$e^{-i(kr - \alpha \ln 2kr - \frac{\ell\pi}{2} + \sigma_\ell)} \tag{2-80}$$

$$+ B_\ell e^{i(kr - \alpha \ln 2kr - \frac{\ell\pi}{2} + \sigma_\ell)}$$

The solution of this equation yields

$$A_\ell = -2i\, e^{i\delta_\ell}$$

$$B_\ell = -e^{2i\delta_\ell} \tag{2-81}$$

$$C_\ell = C_\ell'\, e^{i\delta_\ell} \tag{2-82}$$

In order for (2-78) to reduce to equation (2-76) when V_N and thus δ_N are zero,

$$C_\ell' = \frac{i^\ell}{k}(2\ell+1)e^{i\sigma_\ell} \tag{2-83}$$

and the total wave function becomes

$$\chi^{(+)} = \frac{1}{kr}\sum i^\ell (2\ell+1)\, e^{i\sigma_{T\ell}}\, \mathcal{F}_\ell(kr)\, P_L(\cos\theta) \tag{2-84}$$

where $\sigma_{T\ell} = \sigma_\ell + \delta_\ell$ (2-85)

Using the Legendre addition theorem to write this in terms of the angular coordinates of the vectors $\underset{\sim}{k}$ and $\underset{\sim}{r}$, equation (2-84) becomes

$$\chi^{(+)}(kr) = \frac{4\pi}{kr} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} i^l e^{i\sigma_l} \times \mathcal{F}_l(kr)\, Y^*_{lm}(\theta'\phi')\, Y_{lm}(\theta\phi) \tag{2-86}$$

where the primed variables will be taken as the coordinate corresponding to $\underset{\sim}{k}$.

Inelastic Integral Equation

The following development parallels that used in the study of the general integral equation, the main difference being that distorted waves are used instead of plane waves. The inelastic component of the scattering has already been isolated in equation (2-43) and the corresponding integral equation for   is

$$\Psi_F = -\frac{2m_F}{\hbar^2} \int \mathcal{K}(\underset{\sim}{r}_F, \underset{\sim}{r}_F') \langle v_p | V | \Psi \rangle d^3 r' \tag{2-87}$$

$\mathcal{K}(\underset{\sim}{r}, \underset{\sim}{r}')$, the Green's function is a solution to the following equation. Note that the f subscripts have been dropped as before.

$$(\nabla^2 - \mathcal{U} + k^2)\, \mathcal{K}(\underset{\sim}{r}, \underset{\sim}{r}') = -\delta(\underset{\sim}{r} - \underset{\sim}{r}') \tag{2-88}$$

It was seen in equations (2-21) and (2-22) that the Dirac delta function could be expressed as a series of spherical harmonics as follows

$$\delta(\underline{r}-\underline{r}') = \frac{1}{r^2}\delta(r-r')\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_{lm}^{*}(\theta'\phi') Y_{lm}(\theta\phi)$$

Equation (2-88) differs from (2-16) in the previous treatment only in the addition of a potential term which contains no angular dependence. The series expansion of the Green's function defined by (2-88) will then have the same angular dependence as (2-23) and can be written

$$\mathcal{K}(\underline{r}\,\underline{r}') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \mathcal{G}_l(r,r') Y_{lm}^{*}(\theta'\phi') Y_{lm}(\theta\phi) \qquad (2\text{-}89)$$

When this is inserted into (2-88) using the Laplacian given in (2-24), the radial Green's function is seen to be a solution of the equation

$$\frac{d^2}{dr^2}(r\mathcal{G}_l) + \left(k^2 - \mathcal{U}(r) - \frac{l(l+1)}{r^2}\right) r\mathcal{G}_l = -\frac{1}{r}\delta(r-r') \qquad (2\text{-}90)$$

The reader is referred to equation (2-25) and its subsequent discussion, where more detail is given to a similar calculation.

Letting $\quad \mathcal{G}(r,r') = r\,\mathcal{G}_l(r,r') \qquad (2\text{-}91)$

equation (2-90) becomes

$$\left(\frac{d^2}{dr^2} + k^2 - \mathcal{U}(r) - \frac{\ell(\ell+1)}{r^2}\right) \mathcal{G}_\ell(r,r') = -\frac{1}{r}\delta(r-r') \qquad (2\text{-}92)$$

The radial Green's function is obtained from two independent solutions to the homogeneous part of this equation, one regular at the origin, and the other corresponding to a purely outgoing scattered wave satisfying the boundary condition at infinity. The regular solution $\mathcal{F}_\ell(k,r)$ is obtained from equation (2-77) with its asymptotic form given in (2-79).

Let $\mathcal{H}_\ell(kr)$ be the solution to (2-77) and thus to the homogeneous part of (2-92) with outgoing boundary conditions. It was shown in equations (2-80) and (2-81) that the asymptotic regular solutions could be written in terms of the two irregular exponential solutions

$$e^{\pm i\left(kr - \alpha \ln 2kr - \frac{\ell\pi}{2} + \sigma_{T\ell}\right)}$$

where the plus and minus signs correspond, respectively, to outgoing and incoming waves.

Thus, asymptotically, $\mathcal{H}_\ell(k,r)$ becomes

$$\mathcal{H}_\ell(kr) \rightarrow e^{\pm i\left(kr - \alpha \ln 2kr - \frac{\ell\pi}{2} + \sigma_{T\ell}\right)} \qquad (2\text{-}93)$$

The Green's function is written

$$\mathcal{G}_\ell(r,r') = \begin{cases} \epsilon_1 \mathcal{J}_\ell(k,r) & r<r' \\ \epsilon_2 \mathcal{H}_\ell(kr) & r>r' \end{cases} \tag{2-94}$$

where, analogous to equation (2-31a), the discontinuity in the derivative of $\mathcal{G}(r,r')$ evaluated at r' is written

$$\epsilon_2 \frac{d}{dr}\mathcal{H}_\ell(kr)\Big|_{r=r'} - \epsilon_2 \frac{d}{dr}\mathcal{J}_\ell(kr)\Big|_{r=r'} = -\frac{1}{r'} \tag{2-95}$$

The constants ϵ_1 and ϵ_2 are determined from the Wronskian relation

$$W(\mathcal{J}_\ell, \mathcal{H}_\ell) = -A' \tag{2-96}$$

where A' is a constant. The Wronskian is evaluated using the asymptotic forms of $\mathcal{J}_\ell$ and $\mathcal{H}_\ell$ and found to be

$$\mathcal{J}_\ell \frac{d}{dr}\mathcal{H}_\ell - \mathcal{H}_\ell \frac{d}{dr}\mathcal{J}_\ell = -k \tag{2-97}$$

A comparison of equation (2-97) with (2-95) yields the values of the constants.

$$\begin{aligned} \epsilon_1 &= \frac{1}{kr'} \mathcal{H}_\ell(kr') \\ \epsilon_2 &= \frac{1}{kr} \mathcal{J}_\ell(kr') \end{aligned} \tag{2-98}$$

Using the notation of (2-39), the radial Green's function becomes

$$\mathcal{G}(r,r') = \frac{1}{kr'}\,\mathcal{F}_\ell(kr_<)\,\mathcal{H}_\ell(kr_>) \tag{2-94a}$$

and the total Green's function is obtained from (2-89), (2-91) and (2-94a).

$$\mathcal{K}(r,r') = \frac{1}{kr'r}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\mathcal{F}_\ell(kr_<)\,\mathcal{H}_\ell(kr_>) \times Y^{*}_{\ell m}(\theta'\phi')\,Y_{\ell m}(\theta\phi) \tag{2-89a}$$

This is then used in expression (2-87) to obtain the outgoing scattered wave.

Inelastic Scattering Amplitude

Using (2-87), an expression for Ψ_F in the asymptotic region is sought. For large r, r>>r', $\mathcal{H}_\ell(kr)$ can be replaced by its asymptotic form given in (2-93) and the Green's function written.

$$\mathcal{K}(r_F, r_F') \longrightarrow \frac{1}{4\pi r_F}\, e^{i(k_F r_F - \alpha \ln 2k_F r_F)} \times \frac{4\pi}{k_F r_F}\sum\sum(-i)^\ell e^{i\sigma_{F\ell}}\,\mathcal{F}_\ell\, Y^{*}_{\ell m}\,Y_{\ell m} \tag{2-99}$$

where $\sigma_{F\ell}$ is defined in expression (2-85) and $\underset{\sim}{r}_f$ is taken along the direction of $\underset{\sim}{k}_f$.

Consider the following part of (2-99)

$$\frac{4\pi}{k_F r_P}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(-i)^{\ell}\mathcal{F}_{\ell}\,Y_{\ell m}^{*}\,Y_{\ell m}\,e^{i\sigma_{T\ell}} \tag{2-100}$$

Upon comparison with (2-86), these are recognized as being similar.

Expression (2-100) is identified with the complex conjugate of the "time reversed" optical wave function, and is denoted by $\chi^{(-)*}(kr)$. In the "time reversed" scattering problem, the initial and final channels have exchanged roles and $\chi^{(-)*}(kr)$ is a solution to equation (2-45a) with incoming boundary conditions, where V_N has been replaced by its complex conjugate to simulate flux re-inserted into the beam. The function is perhaps best generated using the previous calculations for $\chi^{(+)}(k,r)$ and the Wigner Relation[13]

$$\chi^{(-)*}(\underset{\sim}{k},\underset{\sim}{r}) = \chi^{(+)}(-\underset{\sim}{k},\underset{\sim}{r}) \tag{2-101}$$

Using equation (2-86),

$$\begin{aligned}\chi^{(-)*}(kr) &= \frac{4\pi}{kr}\sum\sum i^{\ell} e^{i\sigma_{T\ell}}\mathcal{F}_{\ell}(kr) \\ &\times Y_{\ell m}^{*}(\theta\phi)\,Y_{\ell m}(\pi-\theta,\pi+\phi)\end{aligned} \tag{2-102}$$

The parity of the spherical harmonic is[14]

$$Y_{\ell m}(-\Omega) = (-1)^{\ell}\,Y_{\ell m}(\Omega)$$

Hence, equation (2-102) becomes

$$\chi^{(-)*}(\underset{\sim}{k}, \underset{\sim}{r}) = \frac{4\pi}{kr}\sum\sum(-1)^{\ell} e^{i\sigma_{F\ell}} \mathcal{F}_{\ell}(kr) Y^{*}_{\ell m} Y_{\ell m} \qquad (2\text{-}103)$$

which agrees with (2-100)

The Green's function, (2-99), is then re-written incorporating the "time reversed" wave function.

$$\mathcal{K}(r_F, r_F') \rightarrow \frac{1}{4\pi r_F} e^{i(k_F r_F - \alpha \ln 2k_F r_F)} \chi^{(-)*}(k_F r_F) \qquad (2\text{-}99a)$$

Using equations (2-87) and (2-99a), the asymptotic expression for Ψ_F is obtained.

$$\Psi_F \rightarrow \chi_F^{(+)} \delta_{FI} - \frac{e^{i(k_F r_F - \alpha \ln 2k_F r_F)}}{r_F} \frac{m_F}{2\pi\hbar^2} \int \chi^{(-)*} \langle V_F | V | \Psi \rangle d\vec{r} \qquad (2\text{-}104)$$

The first term is the elastic scattering or optical wave function which is a solution to the homogeneous equation (2-45) and is necessary should the initial and final channels coincide. This would be the case of elastic scattering. The inelastic scattering amplitude is the coefficient of the outgoing spherical wave.

$$f_{IN}(\theta\phi) = -\frac{m_F}{2\pi\hbar^2} \langle V_F \chi_F^{(-)} | V | \Psi \rangle \qquad (2\text{-}105)$$

With the inelastic component of the scattering

amplitude separated, the total scattering amplitude can be written, not only as in equation (2-40), but also as a sum of its elastic and inelastic components. Using equations (2-42) and (2-105), the total scattering amplitude is written,

$$f_T(\theta\phi) = -\frac{m_F}{2\pi\hbar^2}\Big[\langle v_F e^{i\underline{k}_F\cdot\underline{r}_F}|U|v_I\chi_I^{(+)}\rangle + \langle v_F\chi_F^{(-)}|V|\Psi\rangle\Big] \tag{2-106}$$

Matrix Element Formalism

Customarily, the scattering amplitude is not used in the literature, but rather the matrix element giving the probability of transition from an initial to some final scattering state. A consideration of the expressions for the scattering amplitude shows the matrix element to be given by

$$T_{FI} = -\frac{2\pi\hbar^2}{m_F} f(\theta\phi) \tag{2-107}$$

where the subscripts i and f refer to the initial and final scattering states, respectively.

In this formalism, equation (2-106) becomes the Gell-Mann-Goldberger relation[15].

$$T_{FI} = \langle v_F e^{i\underline{k}_F\cdot\underline{r}_F}|U|v_I\chi_I^{(+)}\rangle + \langle\chi_F^{(-)}v_F|V|\Psi\rangle \tag{2-108}$$

which, while derived here for the specific case

of the optical wave functions, is quite general and is usually obtained from a unitary transformation of the general scattering amplitude matrix element,

$$T_{FI} = \langle v_F e^{i \underset{\sim}{k}_F \cdot \underset{\sim}{r}_F} | U + V | \Psi \rangle \qquad (2\text{-}109)$$

where U and V are any two potentials that have been isolated.

The angular distribution or cross section is written in this formalism using equation (2-7)

$$\sigma(\theta\phi) = \frac{m_I m_F}{(2\pi\hbar^2)^2} \left(\frac{k_F}{k_I}\right) \sum_{AV} |T_{FI}|^2 \qquad (2\text{-}110)$$

The summation sign indicates a twofold operation; (1) due to the inability of the detector to distinguish between spin orientations, the projections of the final state spins are summed, and (2) the projections of the initial state spins are averaged as when time and thus the direction of particle motion is reversed, the direction of any spin must also be reversed to preserve time reversal invariance.[16] It is seen that if these projections are averaged, producing an "average spin" initial state wave function, the problem is averted. A suitable modification would have to be made, however, if polarization experiments are to be considered.

It now only remains to specify a suitable potential and scattering wave function and, using (2-110), the cross section can be obtained. This problem is considered in the following chapter.

NOTES: CHAPTER II

1 Dicke, Robert H., Wittke, James P., *Introduction to Quantum Mechanics*, (Addison-Wesley, 1960). p. 61.

2 Messiah, Albert, *Quantum Mechanics*, (2 vols.; John Wiley and Sons, 1961), I, p. 372.

3 Baker, Francis Todd, "The Scattering of Zero Spin Particles." Unpublished M.A. Thesis, Miami University, Oxford, Ohio, 1964.

4 Jones, P. B., *The Optical Model in Nuclear and Particle Physics*, (Interscience Publishers, 1963).

5 Wu, T., Ohmura, T., *Quantum Theory of Scattering*, (Prentice Hall, 1962).

6 Margenau, Henery, Murphy, George M., *The Mathematics of Physics and Chemistry*, (D. Van Nostrand Company, 1956), p. 232.

7 Schiff, Leonard I., *Quantum Mechanics*, 2nd ed., (McGraw-Hill, 1955), p. 104.

8 Jackson, John David, *Classical Electrodynamics*, (John Wiley and Sons, 1962), p. 68.

9 *Ibid*., p. 79.

10 *Ibid*., p. 65.

11 National Bureau of Standards, *Handbook of Mathematical Functions*, Applied Mathematics Series 55, (Washington, D.C. : Government Printing Office, 1964), p. 437.

12 Margenau, *The Mathematics of Physics and Chemistry*.

13 Rost, E., Austern., "Inelastic Diffraction Scattering," *Physical Review*, 120(1960), p. 1377.

14 Dicke, *Introduction to Quantum Mechanics*, p. 173.

15 Gell-Mann, M., Goldberger, M. L., "The Formal Theory of Scattering," *Physical Review*, 91(1953), p. 398.

16 Blatt, John M., Weisskopf, Victor F., _Theoretical Nuclear Physics_, (John Wiley and Sons, 1952), p. 525.

CHAPTER III

EVALUATION OF THE DIFFERENTIAL CROSS-SECTION FOR COLLECTIVE EXCITATIONS

Introduction

An exact calculation of the matrix element T_{fi} is at best, a difficult problem. Transforming Schrodinger's equation to its integral equation counterpart has not given its solution, but has only provided a convenient way to obtain the scattering cross section once the wave function has been found. The specific interaction potential is, in general, not known and thus an exact solution for Ψ is not easily found. For example, if the angular momentum transferred in the reaction is non-zero, the interaction potential will not, in general, be a scalar function of position and normal methods for solution of the Schrodinger equation fail. Physically meaningful results, however, may be obtained with appropriate approximations for the wave functions and the interaction potential. It is the purpose of this chapter to discuss the techniques involved, and to show their applicability by fitting experimental angular distributions.

In each of the expressions for T_{fi}, four

integration variables are employed. They are $\underset{\sim}{r}_i$, $\underset{\sim}{r}_f$, $\underset{\sim}{\xi}_i$, and $\underset{\sim}{\xi}_f$ where each is a vector quantity. Undoubtedly, both the initial and final position variables, and the initial and final internal variables are related, but the exact nature of this relation is unknown. It would be convenient to have but one position variable and one internal variable when performing these integrations, hence for want of this exact relation let

$$\begin{aligned} \underset{\sim}{r}_I &= \underset{\sim}{r}_F = r \\ \underset{\sim}{\xi}_I &= \underset{\sim}{\xi}_F = \underset{\sim}{\xi} \end{aligned} \tag{3-1}$$

This is called the zero-range approximation.

The angular distribution is now evaluated using the general scattering matrix element.

$$T_{FI} = \langle V_F e^{i\underset{\sim}{k}_F \cdot \underset{\sim}{r}} | U + V | \Psi \rangle \tag{3-2}$$

It will be assumed that the wave function can be at least partially factored and takes on the form

$$\Psi \simeq V_I(\underset{\sim}{\xi}) \, \mathcal{V}_I^{(+)}(\underset{\sim}{k}_I, \underset{\sim}{r}, \underset{\sim}{\xi}) \tag{3-3}$$

where the wave function $v_i(\underset{\sim}{\xi})$ is the eigen function for the initial nuclear state, and $\mathcal{V}_I^{(+)}(\underset{\sim}{k}_i \underset{\sim}{r}, \underset{\sim}{\xi})$ is a solution to the equation

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + U + V - E\right)\psi_I^{(+)} = 0 \qquad (3\text{-}4)$$

It should be noted that the Hamiltonian describing the internal motions is not present. Hence the ξ variables can be treated as constants and the function $\psi_I^{(+)}$ depends on them only in a parametric sense. T_{fi} is then obtained by a two step calculation; (1) equation (3-3), a function of only the r variable, is solved and the resulting solution used to compute the transition amplitude for a fixed value of ξ,

where $$T_{FI}(\xi) = \langle e^{i\mathbf{k}_F\cdot\mathbf{r}}|U+V|\psi_I^{(+)}\rangle \qquad (3\text{-}5)$$

and (2) this result is multiplied by $v_i(\xi)$ and the projection of their product on the final state f is found.

Physically, the above corresponds to the assumption that the initial nuclear state described by the function $v_i(\xi)$ changes so slowly with respect to the transit time of the projectile, that the scattering wave function is completely determined by the value of $v_i(\xi)$ at the instant of the reaction. This is analogous to "stop motion" photography, where a moving object has its motion frozen on film through the use of a short duration

flash of light. In the time considered, the ξ variables are almost stationary and thus it is seen that the Hamiltonian, $\mathcal{H}(\xi)$, can be neglected in the integral equation leading to (3-2) and in (3-3). These assumptions are most valid when high energy projectiles are used and when the nuclear excited states are collective states. Since the excitation energy must be shared with the fairly large mass of the nucleus, the collective variables exhibit a slow change.

The adiabatic theory (Diffraction Scattering) will now be developed and, using the computed angular distributions, will be compared with experimental alpha particle scattering data.

Diffraction Scattering

The Fraunhofer Approximation

The calculation is most conveniently made in the scattering amplitude formalism; hence (3-5) becomes

$$f(\underset{\sim}{k}_F, k_I, \xi) = -\frac{m}{2\pi\hbar^2}\int d^3r\, e^{-i\underset{\sim}{k}_F\cdot\underset{\sim}{r}}(U+V)\,\Psi_I^{(+)} \qquad (3\text{-}6)$$

Equation (3-3) is multiplied through by $2m/h^2$ and the resulting expression is substituted into (3-6) thus transforming away the dependence on the interaction potential.

$$f(k_F, k_I, \xi) = -\frac{1}{4\pi}\int d^3r\, e^{-i\underset{\sim}{k_F}\cdot\underset{\sim}{r}}(\nabla^2 + k_I^2)\,\psi_I^{(+)} \tag{3-7}$$

where
$$k_I^2 = \frac{2mE}{\hbar^2}$$

This is put into a more convenient form by recognizing that the exponential $e^{-i\underset{\sim}{k_f}\cdot\underset{\sim}{r}}$ is a solution to the wave equation.

$$\nabla^2 e^{-i\underset{\sim}{k_F}\cdot\underset{\sim}{r}} = k_F^2\, e^{-i\underset{\sim}{k_F}\cdot\underset{\sim}{r}}$$

If each side is multiplied by $\psi_I^{(+)}$ and the right and left sides are, respectively, added to and subtracted from the integrand of (3-7), the result will leave the integral unchanged, but allow it to be written in the form

$$f(\underset{\sim}{k_F}, \underset{\sim}{k_I}, \xi) = -\frac{1}{4\pi}\int d^3r\, e^{-i\underset{\sim}{k_F}\cdot\underset{\sim}{r}}(k_I^2 - k_F^2)\,\psi_I^{(+)}$$
$$-\frac{1}{4\pi}\int d^3r\left[e^{-i\underset{\sim}{k_F}\cdot\underset{\sim}{r}}(\nabla^2\psi_I^{(+)}) - (\nabla^2 e^{-i\underset{\sim}{k_F}\cdot\underset{\sim}{r}})\,\psi_I^{(+)}\right] \tag{3-8}$$

Making the adiabatic approximation, $k_i \approx k_f$, the first of these integrals becomes vanishingly small and will be neglected in the remainder of this development. Asymptotically $\psi_I^{(+)}$ can be written

$$\psi_I^{(+)} = e^{i\underset{\sim}{k_I}\cdot\underset{\sim}{r}} + \psi_{SCAT} \tag{3-9}$$

where ψ_{scat} is that component of the wave due to the scattering. This is inserted into the second integral of (3-8) with the result

$$f(k_F, k_I \xi) = -\frac{1}{4\pi}(k_F^2 - k_I^2)\int d^3r\, e^{-i(\underset{\sim}{k}_F - \underset{\sim}{k}_I)\cdot \underset{\sim}{r}}$$
$$-\frac{1}{4\pi}\int d^3r\left[e^{-i\underset{\sim}{k}_F\cdot\underset{\sim}{r}}(\nabla^2\psi_{SCAT}) - (\nabla^2 e^{-i\underset{\sim}{k}_F\cdot\underset{\sim}{r}})\psi_{SCAT}\right] \tag{3-10}$$

The first of these integrals vanishes as in (3-8) when the adiabatic approximation is applied. Green's theorem[1] relating the volume and surface integrals of potential functions is

$$\int_{Vol}(\phi\nabla^2\psi - \psi\nabla^2\phi)d^3r = \int_{SURF}[\phi\nabla\psi - \psi\nabla\phi]\cdot d\underset{\sim}{\sigma} \tag{3-11}$$

If it is assumed that the potential $U + V$ is non-vanishing, only within some finite volume, then the second integral in (3-10) is transformed into the following surface integral

$$f(k_F k_I \xi) = -\frac{1}{4\pi}\int_S\left[e^{-i\underset{\sim}{k}_F\cdot\underset{\sim}{r}}\nabla\psi_{SCAT} - (\nabla e^{-i\underset{\sim}{k}_F\cdot\underset{\sim}{r}})\psi_{SCAT}\right] \tag{3-12}$$

where the surface S is chosen so that the interaction potential vanishes outside.

The impingent particles are assumed to be fairly energetic so that the high energy approximation can be used. In this approximation, the paths of the

incoming particles are assumed to follow straight lines parallel to the vector $\underset{\sim}{k}_i$. The surface S is then conveniently chosen to be a cylinder whose axis lies along the vector $\underset{\sim}{k}_i$ as shown in figure (3-1).

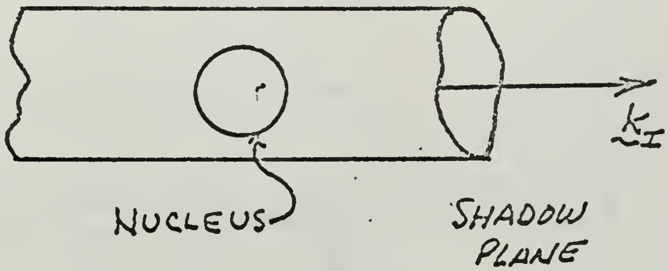


FIGURE (3-1)

With this choice of surface, ψ_{scat} vanishes everywhere except in the shadow plane in the region of the nuclear shadow. The entire contribution to the integral in (3-12) comes from this same region and the element of area $d\sigma$ becomes

$$d\sigma = \hat{k}_I \, dA$$

where $\hat{k}_i$ is a unit vector along $\underset{\sim}{k}_i$ and dA an element of area in the shadow plane. Equation (3-12) is then written

$$f(k_F, k_I \xi) = -\frac{1}{4\pi} \underbrace{\int}_{\text{Shadow}} e^{i \underset{\sim}{k}_F \cdot \underset{\sim}{r}} \left\{ (\hat{k}_I \cdot \nabla) + i (\hat{k}_I \cdot \underset{\sim}{k}_F) \right\} \psi_{scat} \, dA \qquad (3\text{-}13)$$

To compute the Fraunhofer amplitude, the nucleus is assumed perfectly absorbing or "black" to the incident beam and thus $\psi_I^{(+)} = 0$ since there can be no particles in the shadow region. ψ_{scat} is obtained from (3-9) by setting $\psi_I^{(+)} = 0$.

$$\psi_{SCAT} = -e^{i\underset{\sim}{k_I}\cdot\underset{\sim}{r}}$$

This is incorporated in equation (3-11) which becomes,

$$f_{FR} = \frac{k_I}{4\pi}\left(1+\frac{k_F}{k_I}\cos\Theta\right)\int_{SHADOW} e^{i(\underset{\sim}{k_I}-\underset{\sim}{k_F})\cdot\underset{\sim}{r}}\,dA \qquad (3\text{-}14)$$

where Θ, the scattering angle, is the angle between the vectors $\underset{\sim}{k_i}$ and $\underset{\sim}{k_f}$. The vector difference $\underset{\sim}{k_i}-\underset{\sim}{k_f}$ is the momentum transferred to the nucleus. It has magnitude $2k\,\mathrm{Sin}\,(\Theta/2)$ if the adiabatic approximation $k_I \approx k_F \approx k$ is made and for small scattering angles has a direction very nearly perpendicular to $\underset{\sim}{k_i}$. Thus for small Θ, the momentum transfer vector lies in the shadow plane and equation (3-14) can be written

$$f_{FR} = \frac{k}{4\pi}\left(1+\cos\Theta\right)\int_0^{2\pi}\int_0^{R} e^{ikr\sin\frac{\Theta}{2}\cos\phi}\, r\,dr\,d\phi \qquad (3\text{-}15)$$

where the elemental area dA is written in plane polar coordinates, and ϕ is the angle between the momentum transfer vector and $\underset{\sim}{r}$.

The Theory of Blair and Drozdov[2,3]

The collective variables ξ can only affect the scattering amplitude by changing the size and shape of the nuclear surface and thus the nuclear shadow. Hence the collective model proposed by Bohr and Mottelson is appropriate.[4]

It is assumed that the nuclear surface can be described by the following multipole expansion

$$R = R_0 + \sum_{LM} \xi_{LM} Y_{LM}(\theta\,\phi) \tag{3-16}$$

where the ξ_{LM} are the deformation distances of the LM multipoles. If the incident beam direction lies along the z-axis, the deformation of the nuclear equator and thus the corresponding geometrical shadow can be approximated by

$$R = R_0 + \sum_{LM} \xi_{LM} Y_{LM}(\pi/2, \phi) \tag{3-17}$$

This is inserted into the expression for the Fraunhofer amplitude given in (3-15) which becomes

$$f_{FR} = \frac{ik}{4\pi}(1+\cos\Theta) \int_0^{2\pi} \int^{R_0 + \sum_{LM} \xi_{LM} Y_{LM}(\pi/2\,\phi)} e^{i2kr\sin\frac{\Theta}{2}\cos\phi}\, r\,dr\,d\phi \tag{3-18}$$

Equation (3-18) is written as a sum of two integrals, one over a disk of radius R_o and the other over the deformation distances as follows;

$$f_{FR} = \frac{ik}{2}(1+\cos\Theta)\frac{1}{2\pi}\int_0^{2\pi}\int_0^{R_o} e^{i2kr\sin\frac{\Theta}{2}\cos\phi}\, r\,dr\,d\phi$$
$$+ \frac{ik}{2}(1+\cos\Theta)\frac{1}{2\pi}\int_0^{2\pi}\int_0^{R_o+\sum_{LM}\xi_{LM}Y_{LM}(\pi/2,\phi)} e^{i2kr\sin\frac{\Theta}{2}\cos\phi}\, r\,dr\,d\phi \qquad (3\text{-}19)$$

The first of these two integrals gives the elastic scattering component and can be calculated by noting the following integral representation for the Bessel function.[5]

$$2\pi J_m(z)\, i^m = \int_0^{2\pi} e^{iz\cos\phi}\cos m\phi\, d\phi \qquad (3\text{-}20)$$

Using (3-20), it is seen that the ϕ integration gives

$$f_{EL} = \frac{ik}{2}(1+\cos\Theta)\int_0^{R_o} J_0(2kr\sin\tfrac{\Theta}{2})\, r\,dr \qquad (3\text{-}21)$$

performing the r integration gives

$$f_{EL} = \frac{ik}{2}(1+\cos\Theta)\,\frac{J_1(2kR_o\sin\frac{\Theta}{2})}{2kR_o\sin\frac{\Theta}{2}} \qquad (3\text{-}22)$$

Using this result, the elastic angular distribution is calculated from (2-17).

Returning to the second of the two integrals of (3-19), the inelastic scattering amplitude will be obtained where only terms linear in the deformation distances ξ_{LM} will be retained. Integration of the radial variable by parts yields.

$$f_{in} = \frac{ik}{2}(1+\cos\Theta)\int_0^{2\pi} d\phi\, e^{\lambda R_0}\Big\{\Big[\frac{R_0+\sum \xi_{LM}Y_{LM}}{\lambda} - \frac{1}{\lambda^2}\Big]$$

$$\times\, e^{\lambda\sum \xi_{LM}Y_{LM}} - \Big[\frac{R_0}{\lambda} - \frac{1}{\lambda^2}\Big]\Big\} \qquad (3\text{-}23)$$

where $\lambda = i2k\sin\frac{\Theta}{2}\cos\phi$

the exponential becomes, to the first order,

$$e^{\lambda\sum_{LM}\xi_{LM}Y_{LM}} = 1 + \lambda\sum_{LM}\xi_{LM}Y_{LM} \qquad (3\text{-}24)$$

When this is inserted into (3-23), the integral simplifies to the following.

$$f_{IN} = \frac{ikR_0}{2}(1+\cos\Theta)\frac{1}{2\pi}\int_0^{2\pi} d\phi\, e^{i2kR_0\sin\frac{\Theta}{2}\cos\phi}\sum_{LM}\xi_{LM}Y_{LM} \qquad (3\text{-}25)$$

To calculate the integral, the following expression is used[6]

$$Y_{LM}(\pi/2\,\phi) = \begin{cases} \left(\frac{2L+1}{4\pi}\right)^{1/2}\frac{[(L-M)!\,(L+M)!]^{1/2}}{(L-M)!!\,(L+M)!!}(-1)^{\frac{L+M}{2}}e^{iM\phi} & L+M \text{ EVEN} \\ 0 \quad \text{FOR } L+M \text{ ODD} \end{cases} \qquad (3\text{-}26)$$

where $(2n)!! = 2\cdot4\cdot6\cdots(2n)$

and $(2n+1)!! = 1\cdot3\cdot5\cdots(2n+1)$

This is substituted in (3-25), and the order of summation and integrations reversed.

$$f_{lM} = \frac{ikR_0}{2}(1+\cos\Theta)\sum_{LM}\left(\frac{2L+1}{4\pi}\right)^{1/2}\xi_{LM}\frac{[(L-M)!(L+M)!]^{1/2}}{(L-M)!!(L+M)!!}$$
$$\times(-1)^{\frac{L+M}{2}}\frac{1}{2\pi}\int_0^{2\pi}d\phi\, e^{2ikR_0\sin\frac{\Theta}{2}\cos\phi+im\phi} \qquad (3\text{-}27)$$

L + M even

A consideration of the integral shows that it can be written

$$f_{lm} = \frac{ikR_0}{2}(1+\cos\Theta)\sum_{LM}\left(\frac{2L+1}{4\pi}\right)^{1/2}\frac{[(L-M)!(L+M)!]^{1/2}}{(L-M)!!(L+M)!!}$$
$$\times\xi_{LM}(-1)^{\frac{L+M}{2}}i^M J_m\left(2kR_0\sin\frac{\Theta}{2}\right) \qquad (3\text{-}28)$$

When m is an integer, the Bessel functions of order m and -m are not independent and are related by the expression

$$J_m = (-1)^m J_{-m}$$

Hence (3-27) can be written in a more convenient form

$$f_{lm} = \frac{ikR_0}{2}(1+\cos\Theta)\sum\left(\frac{2L+1}{4\pi}\right)^{1/2}\xi_{LM}\frac{[(L-M)!(L+M)!]^{1/2}}{(L-M)!!(L+M)!!} \qquad (3\text{-}28a)$$
$$\times\left(\frac{-iM}{|M|}\right)^L J_{|M|}\left(2kR_0\sin\frac{\Theta}{2}\right)$$

L + M EVEN

The scattering amplitude is then multiplied by

$v_i(\xi)$ and the projection of this product on the final state $v_f(\xi)$ calculated. The resulting amplitude becomes

$$f_{IN} = \frac{ikR_0}{2}(1+\cos\Theta)\sum_{LM}\left(\frac{2L+1}{4\pi}\right)^{1/2}\left(\frac{-iM}{|M|}\right)^L \langle V_f|\xi_{LM}|V_i\rangle$$

$$\times\ \frac{[(L+M)!\,(L-M)!]^{1/2}}{(L-M)!!\,(L+M)!!}\, J_{|M|}\left(2kR_0 \sin\frac{\Theta}{2}\right) \qquad (3\text{-}29)$$

The calculation of the matrix element requires that a nuclear model be specified. In the collective model, the nucleus is represented by a deformed sphere of change. Excited states can be characterized by vibrations and rotations of this spheroid. The low lying states would correspond to rotations about the symmety axis.

The collective model assumes the nucleus to be in a deformed state characterized by a multipole of order L. The radius of the nuclear surface is then

$$R = R_0\left(1+\beta_L Y_{L0}(\alpha)\right) \qquad (3\text{-}30)$$

where α is the polar angle with respect to the symmetry axis and β_L is the deformation parameter corresponding to the L^{th} multipole. By the Legendre addition theorem, this is expanded into the space

fixed coordinate system.

$$R = R_0\left[1 + \beta_L\left(\frac{4\pi}{2L+1}\right)^{1/2} \sum_{M=-L}^{L} Y^*_{LM}(\theta'\phi') Y_{LM}(\theta\phi)\right] \tag{3-31}$$

where the primes indicate the angular coordinates of the symmetry axis. To compute the value of the matrix element, the deformation operators ξ_{LM} must be expressed in terms of the variables used in the collective model. When (3-31) is compared with (3-16), it is seen that for a particular L state,

$$\xi_{LM} = \left(\frac{4\pi}{2L+1}\right)^{1/2} \beta_L R_0 Y^*_{LM}(\theta\phi) \tag{3-32}$$

If it is first assumed that the nucleus is initially in a spin-zero state and then that the angular dependence of the nuclear wave functions can be expressed as spherical harmonics, the calculation of the matrix element is simply

$$\langle LM|\xi_{LM}|00\rangle = \frac{\beta_L R_0}{(2L+1)^{1/2}} \int Y^*_{LM} Y_{LM}\, d\Omega$$

The integral is the orthogonality integral for the spherical harmonics and has the value of unity.

$$\langle V_F|\xi_{LM}|V_I\rangle = \frac{\beta_L R_0}{(2L+1)^{1/2}} \tag{3-33}$$

The angular distribution of particles inelastically scattered from a nucleus initially in a "spin-zero" state and excited to a given multipole L is obtained from equations (2-7), (3-29) and (3-33).

$$\sigma(0\to L) = \frac{(kR_0)^2(\beta_L R_0)^2}{16\pi}\left(1+\cos\Theta\right)^2 \sum_M \frac{(L-M)!\,(L+M)!}{\left[(L-M)!!\,(L+M)!!\right]^2} \times J^2_{|M|}\left(2kR_0 \sin\frac{\Theta}{2}\right) \tag{3-34}$$

$$M = -L,\ -L+2 \cdots$$

This is the basic result of the Blair-Drozdov theory. For the readers convenience, the elastic and inelastic angular distributions for the first four excited states are listed below.

$$\sigma(\Theta) = \frac{k^2R_0^4}{4}\left(1+\cos\Theta\right)^2 \left\{\frac{J_1\left(2kR_0\sin\frac{\Theta}{2}\right)}{2kR_0\sin\frac{\Theta}{2}}\right\}^2$$

$$\sigma(0\to 0) = \frac{(kR_0)^2(\beta_0 R_0)^2}{16\pi}\left(1+\cos\Theta\right)^2 J_0^2\left(2kR_0\sin\frac{\Theta}{2}\right)$$

$$\sigma(0\to 1) = \frac{(kR_0)^2(\beta_1 R_0)^2}{16\pi}\left(1+\cos\Theta\right)^2 J_1^2\left(2kR_0\sin\frac{\Theta}{2}\right)$$

$$\sigma(0\to 2) = \frac{(kR_0)^2(\beta_2 R_0)^2}{16\pi}\left(1+\cos\Theta\right)^2\left\{\frac{1}{4}J_0^2\left(2kR_0\sin\frac{\Theta}{2}\right)+\frac{3}{4}J_2^2\left(2kR_0\sin\frac{\Theta}{2}\right)\right\}$$

$$\sigma(0\to 3) = \frac{(kR_0)^2(\beta_3 R_0)^2}{16\pi}\left(1+\cos\Theta\right)^2\left\{\frac{3}{8}J_1^2\left(2kR_0\sin\frac{\Theta}{2}\right)+\frac{5}{8}J_3^2\left(2kR_0\sin\frac{\Theta}{2}\right)\right\}$$

The characteristic property of equation (3-34) is that cross sections for even values of L are calculated from a linear combination of the squares of even ordered Bessel functions, while those for odd values of L use Bessel functions of odd order. Beyond the first maximum, Bessel functions rapidly approach their asymptotic form.[7]

$$J_L \rightarrow \frac{\sin\left(2kR_o \sin\frac{\Theta}{2} - \frac{\pi}{4} - \frac{L\pi}{2}\right)}{\left(\pi k R_o \sin \Theta/2\right)^{1/2}} \qquad (3\text{-}35)$$

Hence it is seen that the angular distributions for all even parity transitions have the same shape. It follows, then, that all odd parity transitions have the same shape. The two, however, oscillate exactly 180 degrees out of phase with each other. It is further noted that the angular distributions for all odd parity transitions are exactly in phase with the elastic cross section, thus providing a means for parity identification. This rule, known as the Blair phase rule, is usually well obeyed in experiment.

Analysis of Data

A consideration of equation (3-34) shows that there are two adjustable parameters within the context of the theory; R_o, a quantity of the order of the

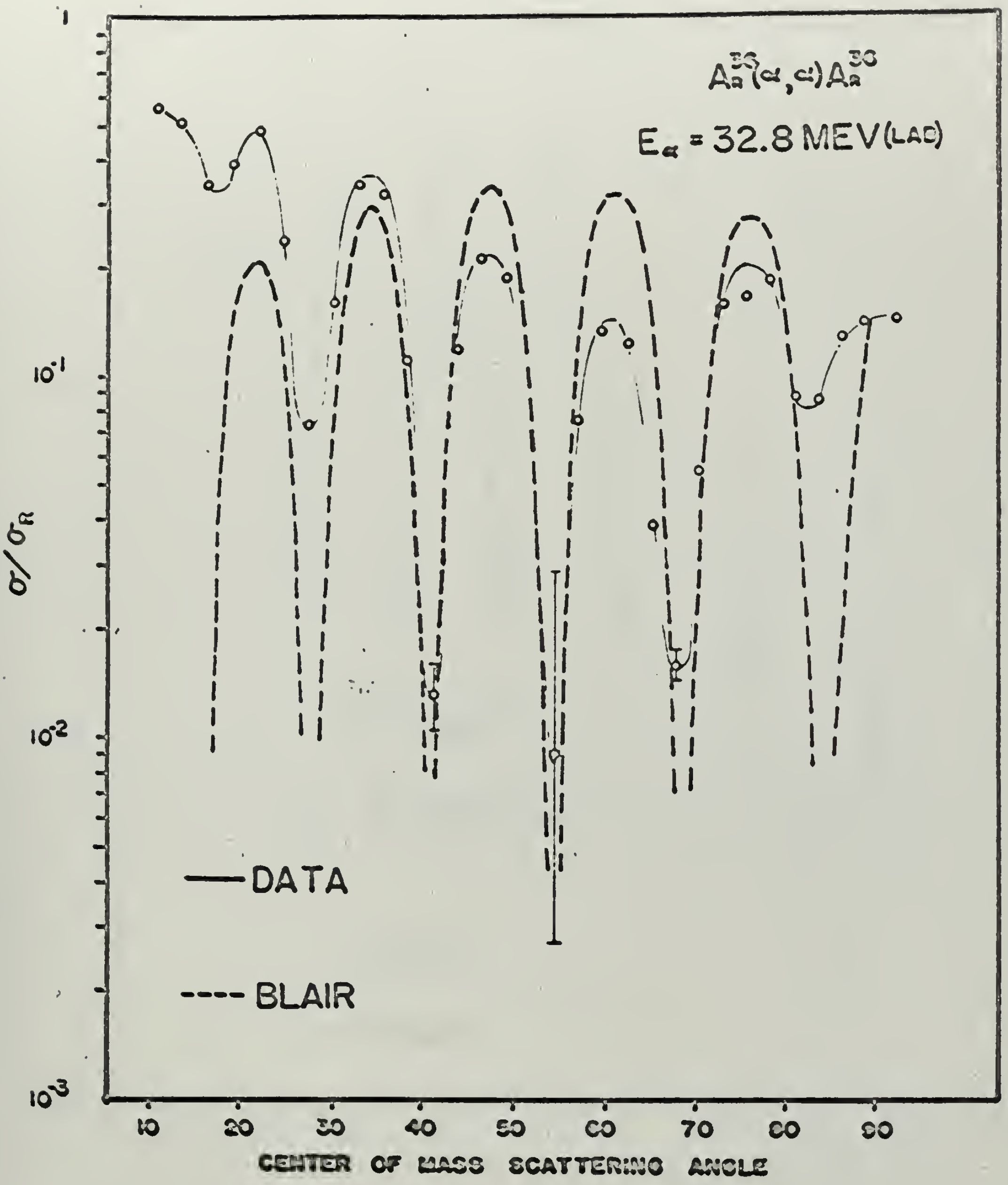
nuclear radius and β_L, the deformation parameter. R_0 is found by fitting equation (3-22) to the elastic cross section.

Fits to $Ar^{36}(\alpha,\alpha)Ar^{36}$ data are shown in figures (3-2) and (3-3) for incident energies of 32.8 and 41.0 Mev(Lab), respectively. R_0 was taken as 6.45 Fermis in both cases. For those angles considered, the Blair theory predicts the oscillatory nature of the cross section, reproducing the maxima and minima very well. It does, however, unlike the experimental distribution, have zeroes. In the back angles, the magnitude of the theoretical prediction drops off more slowly than the data and fits, in general, become much worse.

It should be noted that due to the large variations of the elastic cross section, these figures show plots of the ratio to Rutherford scattering, where the Rutherford cross section[8] is given by

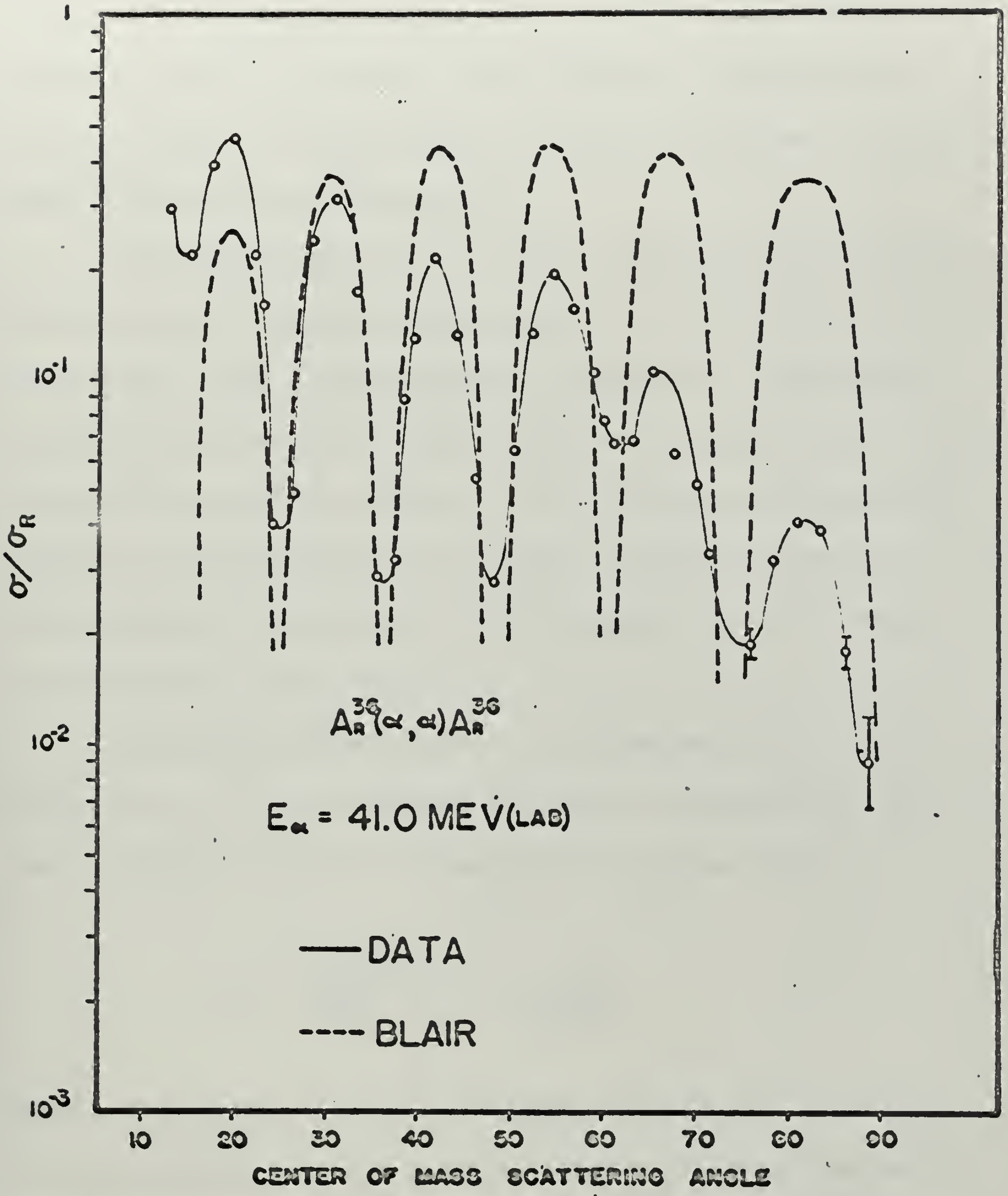
$$\sigma_R = \frac{me^2}{\hbar^2 k^2} \operatorname{Cosec}^4 \frac{\Theta}{2}$$

With the radius parameter R_0 determined, the inelastic distribution may be calculated to within a normalization constant using expression (3-34). The theoretical distribution is then normalized to the experimental cross section and β_L is obtained.



$Ar^{36}(\alpha,\alpha)Ar^{36}$
E_α = 32.8 MEV(LAB)
1
10^{-1}
10^{-2}
10^{-3}
σ/σ_R
—— DATA
---- BLAIR
10
20
30
40
50
60
70
80
90
CENTER OF MASS SCATTERING ANGLE

FIGURE (3-2)



σ/σ_R
1
10^{-1}
10^{-2}
10^{-3}
$A_R^{36}(\alpha,\alpha)A_R^{36}$
E_α = 41.0 MEV(LAB)
—— DATA
---- BLAIR
10
20
30
40
50
60
70
80
90
CENTER OF MASS SCATTERING ANGLE

FIGURE (3-3)

There is some concern, however, as to which peak is to be fitted since it is expected that the Blair formula will be inaccurate at small scattering angles ($\lesssim 20^\circ$) because of its neglect of coulomb effects and at large angles ($\gtrsim 90^\circ$) because of various small angle approximations.

In the derivation, it was assumed that particles having impact parameters less then R_0 would be absorbed, and those with larger impact parameters would pass unaffected, that is they would not be scattered. For charged projectiles this is an obvious oversimplification, as certainly two charged particles passing at distances of the order of the nuclear radius would experience a deflection.

The apsidal distance, or distance of closest approach, d, of a charged particle scattering from the coulomb field of a nucleus is given by[9]

$$d = \frac{Z z e^2}{2E}\left(1 + \csc\frac{\Theta}{2}\right)$$

where Z and z are respectively, the charge numbers of the target and projectile nuclei, E the center of mass energy of the incident projectile and Θ, the center of mass scattering angle. This is solved for Θ, giving,

$$\Theta = 2\sin^{-1}\left[\left(\frac{2Ed}{zZe^2}-1\right)^{-1}\right]$$

If the apsidal distance is taken to be the order of R_o, then for 41 Mev alpha particles incident on Ar^{36}, the scattering angle is of the order of 22 degrees. The corresponding result for the 32.8 Mev alphas is 27 degrees. The "black" nature of the nucleus will not allow impact parameters less then R_o and thus the coulomb scattering should fall off beyond these angles. Hence normalization of the cross sections to the second peak seems justified.

The level scheme[10] for the first three low lying states of Ar^{36} is shown in figure (3-4).

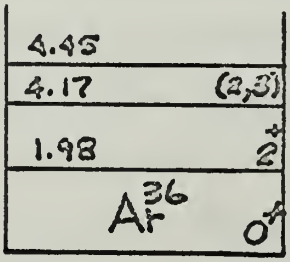


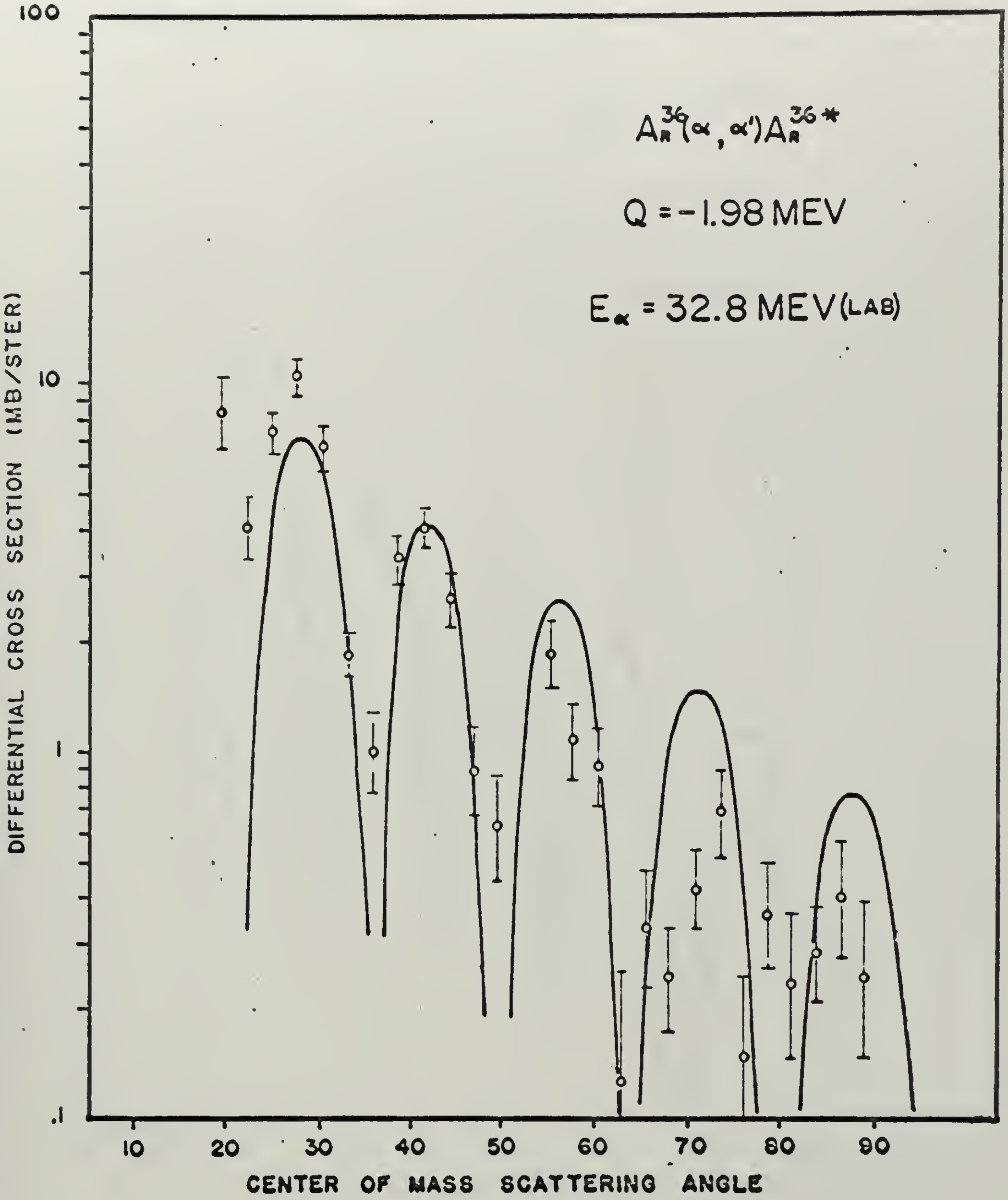
FIGURE (3-4)

Fits to the $Ar^{36}(\alpha,\alpha')Ar^{36*}$ data are shown in figures (3-5) through (3-8) for 32.8 Mev and 41.0 Mev alpha particles. It will be noted that the product $\beta_L R_o$ is more customarily quoted in the literature, and will be used here. The first

excited state is strongly excited and it seen to be a 2^+ state. Hence it is fitted using equation (3-34) with L set equal to 2. The results for the lower energy data are shown in figure (3-5) where $\beta_2 R_0 = .82$ Fermis and for the higher energy data in figure (3-6), where $\beta_2 R_0 = .71$ Fermis. It is interesting to note how well the Blair phase rule is obeyed. A comparison of the L = 2 data with the corresponding elastic data shows them to oscillate exactly 180 degrees out of phase.

The second excited state is strongly excited, with its spin known to be either 2 or 3. Upon comparison of the experimental distributions shown in figures (3-7) and (3-8), with the corresponding elastic data, they are seen to oscillate in phase. If the Blair phase rule is correct, then the second excited state can only be a 3^- state and thus its spin and parity are determined. The angular distributions are calculated using (3-32) with L set equal to 3. These results are also shown in figures (3-7) and (3-8) where $\beta_3 R_0 = .67$ Fermis for the lower energy data, and .56 Fermis for the higher energy data.

The general appearance of the inelastic angular distributions is similar to their elastic counterparts. The oscillatory nature is predicted and the position of the maxima and minima are reproduced well.



$A_R^{36}(\alpha,\alpha')A_R^{36*}$
Q = -1.98 MEV
E_α = 32.8 MEV(LAB)
DIFFERENTIAL CROSS SECTION (MB/STER)
100
10
1
.1
10
20
30
40
50
60
70
80
90
CENTER OF MASS SCATTERING ANGLE

FIGURE (3-5)

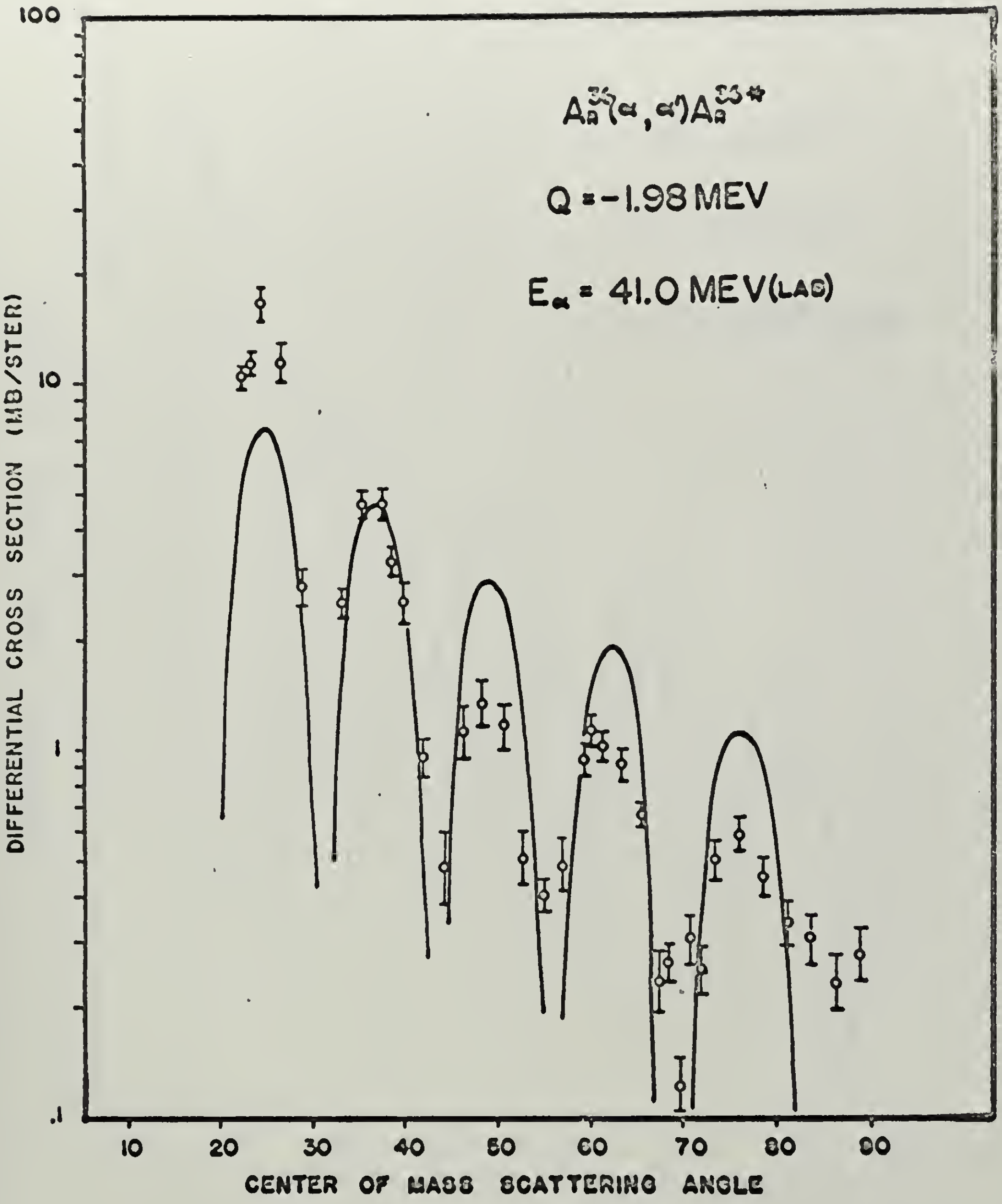

$A_{18}^{36}(\alpha,\alpha')A_{18}^{36*}$
Q = -1.98 MEV
E_α = 41.0 MEV(LAB)
DIFFERENTIAL CROSS SECTION (MB/STER)
100
10
1
.1
10
20
30
40
50
60
70
80
90
CENTER OF MASS SCATTERING ANGLE

FIGURE (3-6)

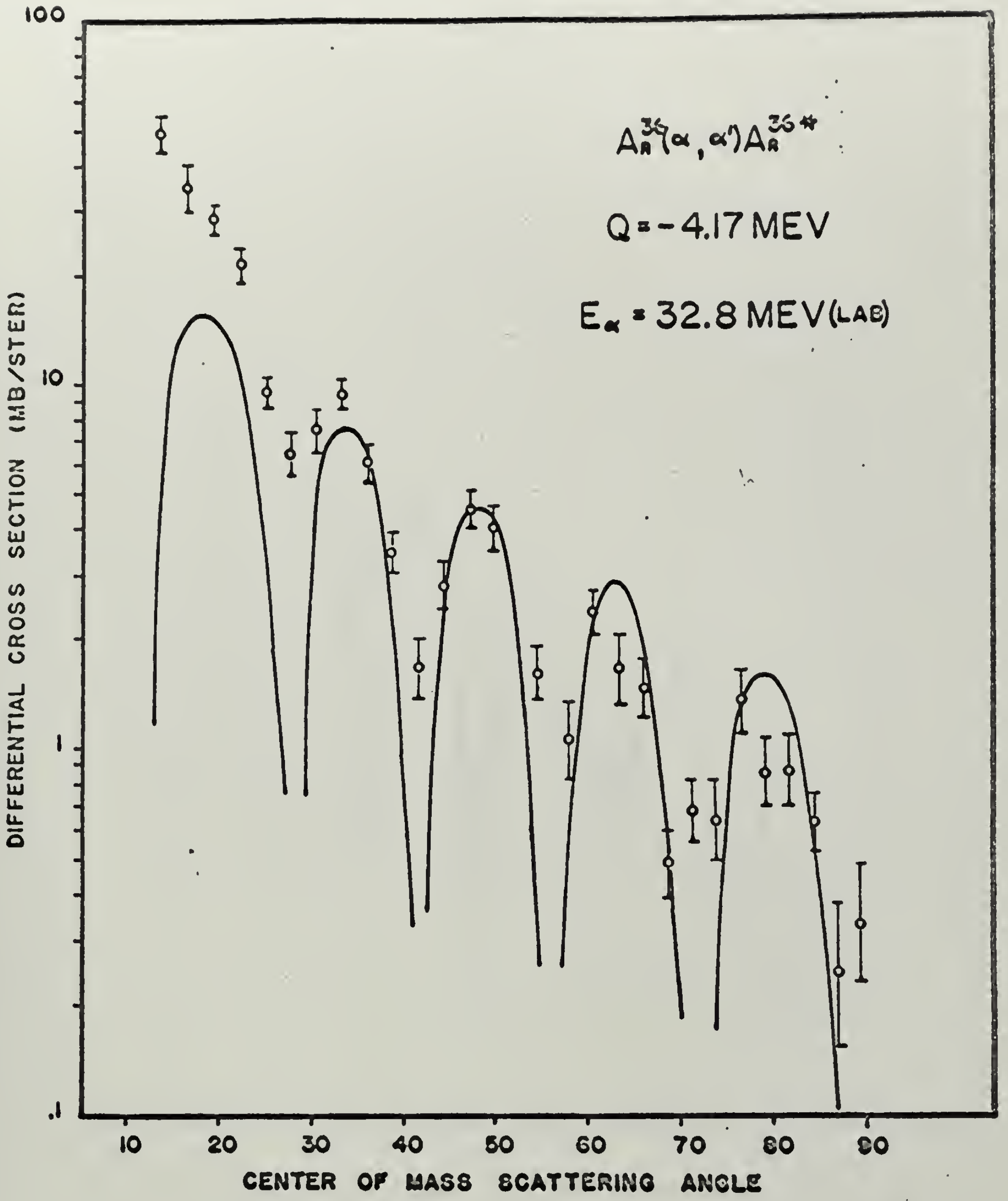

100
10
1
.1
DIFFERENTIAL CROSS SECTION (MB/STER)
A$_R^{36}$(α, α')A$_R^{36*}$
Q = -4.17 MEV
E$_\alpha$ = 32.8 MEV(LAB)
10
20
30
40
50
60
70
80
90
CENTER OF MASS SCATTERING ANGLE

FIGURE (3-7)

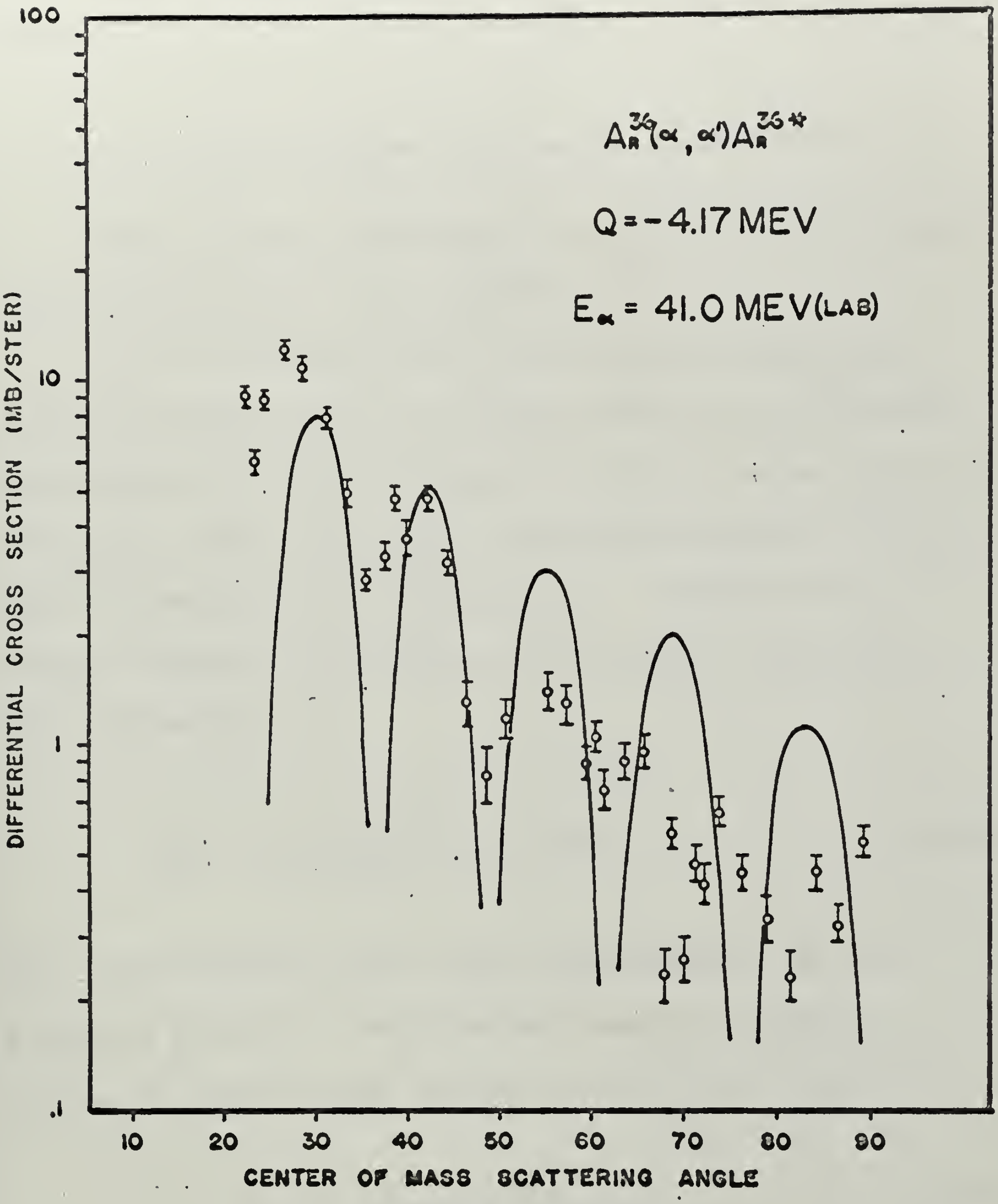

$A_R^{36}(\alpha,\alpha')A_R^{36*}$
Q = -4.17 MEV
E_α = 41.0 MEV(LAB)
100
10
1
.1
DIFFERENTIAL CROSS SECTION (MB/STER)
10
20
30
40
50
60
70
80
90
CENTER OF MASS SCATTERING ANGLE

FIGURE (3-8)

As in the elastic case, the inelastic distribution predicts zero minima and fares poorly in the back angles.

The Distorted Wave Born Approximation

This and the remaining text follow the treatments given by Austern[11] and McCarthy.[12]

If it is assumed that the average interaction potential $\mathcal{U}(r)$ of the colliding particle is primarily responsible for determining the optical wave functions $\chi(\underset{\sim}{k},\underset{\sim}{r})$, and if $k_i \approx k_f$, then the potential $\mathcal{V}(\underset{\sim}{r},\xi)$ may be treated as a perturbation. Consider the matrix element T_{fi} giving the inelastic component of the scattering.

$$T_{FI} = \langle \psi_F \chi_F^{(-)} | V | \Psi \rangle \tag{3-36}$$

The wave function Ψ is then approximated by the outgoing optical wave function that was used in (2-41) to compute the elastic scattering component.

$$\Psi \approx \psi_I(\xi)\chi_I^{(+)}(\underset{\sim}{k}_I,r) \tag{3-37}$$

where the approximation of (3-1) has been applied. With this expression for Ψ, equation (3-36) becomes

$$T_{FI} = \int d^3r\, \chi^{(-)*} \langle v_F | V | v_I \rangle \chi_I^{(+)} \tag{3-38}$$

The width factor $\langle v_f | V_f | v_i \rangle$ influences the magnitude of the cross section and depends on a detailed knowledge of both the reaction processes and the nuclear wave functions. The distorted wave functions $\chi(\underset{\sim}{k},\underset{\sim}{r})$, are primarily responsible for determining the angular distribution.

The potential $V_f(\underset{\sim}{r},\underset{\sim}{\xi})$, assumed a scalar function of position, is expanded in multipoles of the vector $\underset{\sim}{r}$. This expansion becomes

$$V(\underset{\sim}{r},\underset{\sim}{\xi}) = \sum_{LM} V_{LM}(r,\xi)\, Y^*_{LM}(\theta\phi) \tag{3-39}$$

where it is necessary for the ensuing calculations that V_{LM} behave like the spherical harmonics under rotations of the coordinate system and have parity $(-1)^L$. The factor $\langle v_f | V_f | v_i \rangle$ can then be written.

$$\langle v_F | V | v_I \rangle = \sum_{LM} \langle v_F | V_{LM} | v_I \rangle Y^*_{LM} \tag{3-40}$$

The wave functions v_f and v_i are assumed to have their angular dependence described by the spherical harmonics. They will, in general, be a

function of their angular momentum $J\hbar$ and its projection $M\hbar$. A fundamental property of the matrix elements of these spherical harmonics and functions which transform in a like manner under a coordinate rotation, is the Wigner-Echart theorem.[13]

Applied to the above "width" factor, it is

$$\langle V_{J_F M_F} | V_{LM} | V_{J_I M_I} \rangle = (J_I L M_I M | J_F M_F) \langle V_{J_F} \| V_L \| V_{J_I} \rangle \qquad (3\text{-}41)$$

J_i and J_f are the initial and final nuclear spins with M_i and M_f their respective projections. The factor $\langle v_{J_f} \| V_L \| v_{J_i} \rangle$ is the "reduced width" and $(J_i L\, M_i M \mid J_f M_f)$ a Clebsch-Gordan coefficient. The reduced width is just the width factor expressed in a diagonalized representation. Using (3-41), equation (3-40) becomes

$$\langle V_F | V | V_I \rangle = \sum_{LM} (J_I L M_I M | J_F M_F) \langle V_{J_F} \| V_L \| V_{J_I} \rangle Y_{LM}^{*} \qquad (3\text{-}42)$$

The interpretation being that the L^{th} multipole in equation (3-39) transfers an amount of angular momentum L to the nucleus. The Clebsch-Gordan coefficients are so designed as to limit L to the range

$$|\underset{\sim}{J_I} - \underset{\sim}{J_F}| \leq L \leq |\underset{\sim}{J_I} + \underset{\sim}{J_F}|$$

and thus insure conservation of angular momentum. The "reduced width" factor is a function of only the

r variable and it becomes convenient to write it as the product of a "strength factor" and a "form factor" which gives its radial dependence.

$$\langle V_{J_F} \| V_L \| V \rangle = A_L F_L(r) \qquad (3\text{-}43)$$

The strength factor A_L would give the magnitude of the reaction and requires a specific knowledge of both the nuclear wave functions and the nature of the reaction. The form factor, however, need not be known exactly, and meaningful results are obtained from knowing its general trend as will be seen later.

Upon substitution of expressions (3-42) and (3-43) into (3-38) the matrix element becomes

$$T_{FI} = \sum_{LM} A_L (J_I L M_I M | J_F M_F) \tau_{LM}$$

where (3-44)

$$\tau_{LM} = \int d^3r\, \chi_F^{(-)*} \left[Y_{LM}^{*}(\theta\phi)\, F_L(r) \right] \chi_I^{(+)}$$

The cross section is then obtained from expression (2-110),

$$\sigma(\theta) = \frac{m_I m_F}{(2\pi\hbar^2)^2} \left(\frac{k_F}{k_I}\right) \sum_{AV} \Big| \sum_{LM} A_L (J_I L M_I M | J_F M_F) \tau_{LM} \Big|^2 \qquad (3\text{-}45)$$

It will be recalled that the summation sign referred to a sum over M_f and an average of M_i. For an initial state having spin J_i, there are $(2J_i+1)$ values

of M_i, hence (3-45) becomes

$$\sigma(\Theta) = \frac{m_I m_F}{(2\pi\hbar^2)^2}\left(\frac{k_F}{k_I}\right)\frac{1}{2J_I+1}\sum_{M_I}\sum_{M_F}\sum_{LM}\sum_{L'M'}|A_L|^2 \times (J_I L M_I M | J_F M_F)(J_I L' M_I M' | J_F M_F)\,|\tau_{LM}|^2 \tag{3-46}$$

Noting the following Clebsch-Gordan symmetry relationship,[14]

$$(J_I L M_I M | J_F M_F) = (-1)^{L-J_F-M_I}\left(\frac{2J_F+1}{2L+1}\right)^{1/2}(J_I J_F - M_I M_F | LM) \tag{3-47}$$

the cross section can be written

$$\sigma(\Theta) = \frac{m_I m_F}{(2\pi\hbar^2)^2}\left(\frac{k_F}{k_I}\right)\frac{(2J_F+1)}{(2J_I+1)}\sum_{L,M}\sum_{L'M'}|A_L|^2|\tau_{LM}|^2 \times \frac{(-1)^{L+L'}}{(2L'+1)^{1/2}(2L+1)^{1/2}}\sum_{J_I J_F}(J_I J_F - M_I M_F | LM)(J_I J_F - M_I M_F | L'M') \tag{3-48}$$

The Clebsch-Gordan coefficients exhibit the following ORTHOGONALITY ~~closure~~ relationship.[15]

$$\sum_{M_I M_F}(J_I J_F - M_I M_F | LM)(J_I J_F - M_I M_F | L'M') = \delta_{LL'}\,\delta_{MM'} \tag{3-49}$$

Hence the cross section expression simplifies and is written

$$\sigma = \frac{m_I m_F}{(2\pi\hbar^2)^2}\left(\frac{k_F}{k_I}\right)\left(\frac{2J_F+1}{2J_I+1}\right)\sum_{L=0}^{\infty}\frac{|A_L|^2}{(2L+1)}\sum_{M=-L}^{L}|\tau_{LM}|^2 \tag{3-50}$$

If the spin of the initial state is zero, then the

final state spin is just the angular momentum transferred in the reaction. It should be noticed that different values of the momentum transfer L do not interfere in determining the reaction.

The problem of evaluating the cross section has now reduced to that of calculating the matrix element τ_{LM}. Here explicit forms of the optical wave functions are required. These may be approximations[16,17] or with complete rigor, the spherical harmonic expansions derived in chapter II may be used. The following sections discuss the nature of possible approximations.

Plane Waves

While it may seem paradoxical to include a section with this title in a discussion of the distorted wave Born approximation, the use of plane waves is instructive as it gives insight into the nature of the distortion necessary to reproduce the form of the experimental angular distributions.

It is convenient to make the identification

$$\rho(r)\, e^{iS(r)} = \chi_F^{(-)*} \chi_I^{(+)} \qquad (3\text{-}51)$$

with the matrix element τ_{LM} given in (3-45) then becoming

$$\tau_{LM} = \int \rho(r) e^{iS(r)} F_L(r)\, Y^{*}_{LM}(\theta\phi)\, d^3r \qquad (3\text{-}52)$$

The quantity $\rho(r)\, e^{iS(r)}$ is a probability density and may be thought of as describing the properties of a probe which measures the evolution of the nuclear state during the process of inelastic scattering. To clarify this idea, consider the case where the optical wave functions have been replaced by plane waves.

$$\rho(r) e^{iS(r)} = \rho_0\, e^{i \underset{\sim}{k} \cdot \underset{\sim}{r}} \qquad (3\text{-}53)$$

where $\underset{\sim}{K} = \underset{\sim}{k_i} - \underset{\sim}{k_f}$ and ρ_0 is a constant. Here the probe measures the difference in the wave number vectors for the initial and final states and the matrix element τ_{LM} is proportional to the probability that the initial and final states differ by momentum $K\hbar$. Since the inelastic scattering cross section is obtained from this matrix element, the use of plane waves implies that the entire momentum transferred goes into the inelastic component of the scattering.

If $\rho(\underset{\sim}{r})$ is anisotropic in space, the probe tries to make a simultaneous measurement of its position and the momentum transfer. By the uncertainty principle, the resolution of the momentum transfer measurement

must then be reduced. In Butler's[18] theory of nuclear reactions, it is assumed that the reaction is confined to the surface. It would then seem that the probe could obtain no information whatsoever about the radial component of the momentum transfer. On the other hand, since no angular localization is assumed on the nuclear surface, the resolution of the angular momentum transfer should be a maximum. This is seen in the following calculation.

Setting the constant ρ_0 equal to unity, the matrix element τ_{LM} becomes

$$\tau_{LM} = \int e^{i\underline{k}\cdot\underline{r}}\, Y_{LM}^{*}(\theta\phi)\, \delta(r-R_0)\, \frac{d^3r}{r^2} \qquad (3\text{-}54)$$

where F(r) becomes a Dirac delta function limiting the reaction to a spherical shell of radius R_0. Using equation (2-18), the plane wave is expanded in a series of spherical harmonics and (3-54) becomes

$$\tau_{LM} = \sum_{L'M'} i^{L'} j_{L'}(kR_0)\, Y_{L'M'}^{*}(\theta'\phi') \int Y_{LM}^{*}(\theta\phi)\, Y_{L'M'}(\theta\phi)\, d\Omega \qquad (3\text{-}55)$$

The integral in (3-55) is the orthogonality integral for the spherical harmonics where

$$\int Y_{L'M'} Y_{LM}^{*}\, d\Omega = \delta_{LL'}\,\delta_{MM'} \qquad (3\text{-}56)$$

The Kronecker delta functions are zero unless L' and M' equal L and M respectively, hence only the LM term remains.

Equation (3-55) becomes

$$\tau_{LM} = j_L(kR_0)\,Y^{*}_{LM}(\theta'\phi') \tag{3-57}$$

where the primes indicate the angular coordinates of the symmetry axis. As it will be assumed that the nucleus is initially in a spin zero state, the cross section for a given final state L is given by (3-50) to be proportional to

$$\sigma_L \propto |j_L(kR_0)|^2 \sum_{M=-L}^{L} Y^{*}_{LM}(\theta'\phi')\,Y_{LM}(\theta'\phi') \tag{3-58}$$

The Legendre addition theorem is used to convert the sum of spherical harmonics into a Legendre polynomial of argument unity which has magnitude unity for all values of L. Equation (3-58) becomes

$$\sigma_L \propto j_L^2(kR_0) \tag{3-59}$$

Beyond their first maximum, these spherical Bessel functions rapidly approach their asymptotic form given in equation (2-19), and thus in this region, the cross section can be written,

$$\sigma_L \longrightarrow \frac{\sin^2(kR_0 - L\pi/2)}{(kR_0)^2} \tag{3-60}$$

Hence this behaves like the Blair formula. It appears then, that the sharpness of the maxima and the fact that the minima actually reach zero can be associated with the maximum resolution of the angular momentum transfer. If this interpretation is correct, it should be possible to spoil the resolution by causing $\rho(\underline{r})$ to be non-uniform over the nuclear surface.

We proceed by localizing the reaction to part of the nuclear surface. As alpha particles are strongly absorbed, those incident on the nuclear surface will have their wave funcions considerably reduced on the shadow side of the nucleus as compared with the illuminated side. The same effect would be true of the scattered particles. The overlap of their wave functions would be greatest on the side of the nucleus opposite to the direction of momentum transfer as is shown in figure (3-9). It is assumed that that part of the surface axially symmetric about the direction of momentum transfer and cut off by a cone of semiangle α contributes uniformly to the matrix element, and the remaining part of the surface

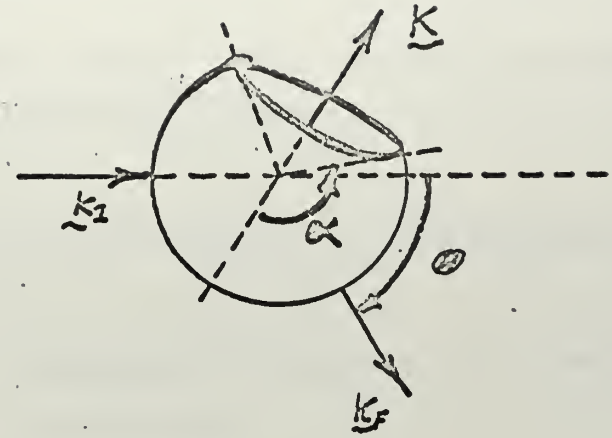


FIGURE (3-9)

contributes nothing at all. The matrix element τ_{LM} is calculated using the expression in (3-54) where the plane wave has been expanded in a series of Legendre polynomials. After the radial integration, (3-54) becomes

$$\tau_{LM} \propto \sum_{L'} i^{L'}(2L'+1)\, j_{L'}(kR_0) \times \int_0^{2\pi}\int_0^{\alpha} Y_{LM}^{*}(\theta\phi)\, P_{L'}(\cos\theta)\sin\theta\, d\theta\, d\phi \qquad (3\text{-}61)$$

where the axis of symmetry is taken along the direction of momentum transfer. The Legendre polynomials and spherical harmonics are related through the expression

$$P_{L'}(\cos\theta) = \sqrt{\frac{4\pi}{2L'+1}}\; Y_{L'0}(\theta\phi) \qquad (3\text{-}62)$$

and as the azimuthal integration has been unaffected,

it is seen by (3-56) that the integral is non-zero only when M equals zero. Using this result, the matrix element τ_{LM} becomes

$$\tau_{LM} \propto \sum_{L'} i^{L'}(2L'+1)^{1/2} j_{L'}(kR_0) \int_0^{\alpha} Y_{L0}^{*}(\theta\phi) Y_{L'0}(\theta\phi) \sin\theta d\theta \quad (3\text{-}63)$$

The integrand is expanded in a series of Legendre polynomials.

$$\begin{aligned} Y_{L0}(\theta\phi) Y_{L'0}(\theta\phi) &= \sum_{\ell} C^2_{LL'\ell} P_\ell(\cos\theta) \\ &= \sum_{\ell} C^2_{LL'\ell} \sqrt{\frac{4\pi}{2\ell+1}} Y_{\ell 0}(\theta\phi) \end{aligned} \quad (3\text{-}64)$$

The expansion coefficient is obtained by multiplying both sides of (3-64) by $Y_{\ell' 0}(\theta,\phi)d\Omega$ and integrating over 4π radians. The orthogonality of the spherical harmonics determines the coefficient $C_{LL'\ell'}$ on the right thus giving the expression

$$C^2_{LL'\ell'} = \iint Y_{L0} Y_{L'0} Y_{\ell' 0} d\Omega \quad (3\text{-}65)$$

The following relation involving Clebsch-Gordan coefficients is noted.[19]

$$\int Y_{L_1M_1} Y_{L_2M_2} Y^{*}_{L_3M_3} d\Omega = \left[\frac{(2L_1+1)(2L_2+1)}{(2L_3+1)}\right]^{1/2} \quad (3\text{-}66)$$

$$\times (L_1 L_2 M_1 M_2 | L_3 M_3)(L_1 L_2 0 0 | L_3 0)$$

Using this relation, equation (3-65) becomes

$$C_{LL'\ell'} = \frac{\left[(2L'+1)(2L+1)\right]^{1/2}}{\left[(2\ell+1)4\pi\right]^{1/2}} (LL'00|\ell 0)^2 \qquad (3\text{-}67)$$

The expansion in (3-64) is then obtained.

$$Y_{L0} Y_{L'0} = \frac{\left[(2L+1)(2L'+1)\right]^{1/2}}{4\pi} \sum_{\ell=0} (LL'00|\ell 0)^2 P_\ell(\cos\Theta) \qquad (3\text{-}68)$$

The matrix element τ_{LM} becomes

$$\tau_{LM} \propto \sum_{L'\ell} i^{L'}(2L'+1) j_{L'}(kR_0)(LL'00|\ell 0)^2 \times \int_0^\alpha P_\ell(\cos\Theta)\sin\Theta\, d\Theta \qquad (3\text{-}69)$$

Hence for a given angular momentum transfer L, the matrix element is proportional to a sum of spherical Bessel functions of different orders. In general this will be complex and it is therefore highly improbable that τ_{LM} will go to zero for any real value of K. The angular resolution associated with zero minima has been spoiled. If $\alpha = \pi$, the entire surface then contributes uniformly. The integral in (3-69) becomes the orthogonality integral for the Legendre polynomials, giving zero unless $\ell = 0$. With $\ell = 0$, the Clebsch-Gordan coefficient will be non-zero only when $L' = L$, hence it is seen that the result reduces to that in equation (3-59).

An interesting interpretation of (3-69) is that for total angular momentum transfer L, L' represents that going into the inelastic component of the scattering as it results from the L'th partial wave in the expansion of a plane wave, and ℓ that which has gone into other components, such as elastic scattering or absorption from the beam.

Approximate Optical Wave Functions

Based on a study of optical model wave functions,[20,21] reasonable approximations[22] for the incoming and outgoing wave functions are found to be

$$\begin{aligned} \chi_{I}^{(+)} &\simeq A(r)\, e^{i\underline{k}_I\cdot\underline{r} - \gamma(\hat{k}_I\cdot\hat{r})} \\ \chi_{F}^{(-)*} &\simeq A(r)\, e^{-i\underline{k}_F\cdot\underline{r} + \gamma(\hat{k}_F\cdot\hat{r})} \end{aligned} \qquad (3\text{-}70)$$

where γ is referred to as the anisotropy parameter and is a measure of the non uniformity of the nuclear surface. When these expressions are used in equations (3-51) and (3-52), the matrix element τ_{LM} is seen to be

$$\tau_{LM} = \int Y_{LM}(\theta\phi)\, F_L(r)\, |A(r)|^2 e^{i\underline{K}\cdot\underline{r} - \frac{\gamma}{r}(\hat{\underline{k}}_I - \hat{\underline{k}}_F)\cdot\underline{r}} \qquad (3\text{-}71)$$

Where as usual, $\underline{K} = \underline{k}_I - \underline{k}_F$. Let,

$$\underset{\sim}{\zeta} = \underset{\sim}{k} + \frac{i\gamma}{r}(\hat{k}_\gamma - \hat{k}_p)$$
$$\zeta = (\underset{\sim}{\zeta}\cdot\underset{\sim}{\zeta})^{1/2} \tag{3-72}$$

and expand the resulting plane wave in a series of spherical harmonics. The angle integrations were computed in equations (3-55) and (3-56) and they result in the following expression for τ_{LM}.

$$\tau_{LM} = 4\pi i^L Y_{LM}^*(\theta_\zeta, \phi_\zeta)\int_0^\infty F_L(r)|A(r)|^2 j_L(\zeta r) r^2 dr \tag{3-73}$$

The cross section is calculated using expression (3-50), where the spin of the initial state is assumed to be zero.

$$\sigma_L \propto \sum_{-L}^{L} Y_{LM}^*(\theta_\zeta \phi_\zeta) Y(\theta_\zeta^* \phi_\zeta^*) \times \left|\int_0^\infty F_L(r)|A_L(r)|^2 j_L(\zeta r) r^2 dr\right|^2 \tag{3-74}$$

Using the Legendre summation formula,

$$4\pi\sum_{-L}^{L} Y_{LM}^*(\theta_\zeta \phi_\zeta)\, Y_{LM}(\theta_\zeta^* \phi_\zeta^*) = (2L+1)\, P_L(\cos\Gamma) \tag{3-75}$$

where Γ is the angle between $\underset{\sim}{\zeta}$ and $\underset{\sim}{\zeta}^*$.
$\cos(\Gamma)$ can be written

$$\cos(\Gamma) = \frac{\underset{\sim}{\zeta}\cdot\underset{\sim}{\zeta}^*}{\zeta\,\zeta^*} \tag{3-76}$$

If the adiabatic approximation $k_i \approx k_f \approx k$, is employed, $\underset{\sim}{\zeta}$ can be written

$$\underset{\sim}{\zeta} \approx 2\left(k + \frac{i\gamma}{r}\right) \sin\frac{\Theta}{2}\ \hat{\underset{\sim}{K}} \tag{3-77}$$

where Θ is the angle between $\underset{\sim}{k_i}$ and $\underset{\sim}{k_f}$ and $\hat{\underset{\sim}{K}}$ is a unit vector along the direction of momentum transfer. (3-75) is inserted in (3-74) and it can be shown that $\text{Cos}(\Gamma) \approx 1$. Hence $P_L(\text{Cos}\,\Gamma) \approx 1$ and the cross section for a given value of L is proportional to

$$\sigma_L \propto \left| \int_0^\infty F_L(r)\,|A(r)|^2 j_L(\zeta r)\, r^2 dr \right|^2 \tag{3-78}$$

It now remains to make some approximation for the radial integral. Since alpha particles are strongly absorbed by the nucleus, $|A(r)|^2$ will fall off for r less than the nuclear radius. On the other hand, F(r) being associated with the nuclear wave function, will fall off for r greater than the nuclear radius. Assuming that F(r) is slowly varying over the mean path of the alpha particle, the radial integration should essentially depend on the radius R_o and the thickness of the surface. A simple empirical choice which involves only these two parameters is

$$F_L(r)\,|A(r)|^2 \propto e^{-\frac{(r-R_o)^2}{\lambda^2}} \tag{3-79}$$

The final form for the differential cross section is therefore

$$\sigma_L \propto \left| \int_0^\infty e^{-\left(\frac{r-R_0}{\lambda}\right)^2} j_L(\xi r)\, r^2 dr \right|^2 \qquad (3\text{-}80)$$

With the adiabatic approximation for ξ given in (3-77), (3-80) can be written in a simpler form.

$$\sigma_L \propto \left| \int_0^\infty e^{-\left(\frac{r-R_0}{\lambda}\right)^2} j_L\left[2(kr+i\gamma)\sin\frac{\Theta}{2}\right] r^2 dr \right|^2 \qquad (3\text{-}81)$$

The corresponding surface interaction result is

$$\sigma_L \propto \left| j_L\left[2(kR_0+i\gamma)\sin\frac{\Theta}{2}\right] \right|^2 \qquad (3\text{-}82)$$

In the discussion resulting in equation (3-69) it was found that by introducing an angular anisotrophy into the reaction, the resolution for a measurement of the angular momentum transferred was at least partially spoiled.

It was seen that the matrix element, τ_{LM}, was no longer proportional to a single spherical Bessel function of order L, but consisted of a sum of $j_{L'}(kR)$ The result in (3-82) is but a single spherical Bessel function. However its argument is complex. That this actually corresponds to (3-69) is shown by expanding (3-82) in a series of $j_{L'}\,(2KR_0 \sin\frac{\Theta}{2})$.

Consider the plane wave expansion, where the magnitude of $K R_o = x + y$

$$e^{i(x+y)\cos\theta} = \sum_n i^n (2n+1) j_n(x+y) P_n(\cos\theta) \tag{3-83}$$

This can also be written

$$e^{ix\cos\theta} e^{iy\cos\theta} = \sum_{L',\ell} i^{L'+\ell} (2L'+1)(2\ell+1) \times j_{L'}(x) j_\ell(y) P_{L'}(\cos\theta) P_\ell(\cos\theta) \tag{3-84}$$

The two series are equated and both sides multiplied by $P_L(\cos\theta) \sin\theta d\theta$. When the result is integrated from 0 to π, orthogonality eliminates all but the L'^{th} term in (3-83) giving the following expression

$$j_L(x+y) = \sum_{L',\ell} i^{L'+\ell-L} (2L'+1)(2\ell+1) j_{L'}(x) j_\ell(y) \times \frac{1}{2} \int P_L(\cos\theta) P_{L'}(\cos\theta) P_\ell(\cos\theta) \sin\theta d\theta \tag{3-85}$$

The integral is written in terms of spherical harmonics and both sides are multiplied by $d\phi$ and integrated from 0 to 2π

$$j_L(x+y) = \sum_{L'\ell} (i^{L'+\ell-L})(2L'+1)(2\ell+1) j_{L'}(x) j_\ell(y) \times \left(\frac{4\pi}{(2L+1)(2\ell+1)(2L'+1)} \right)^{1/2} \int Y_{L0} Y_{\ell' 0} Y_{\ell 0} d\Omega \tag{3-86}$$

The integral is given in (3-66), and with the proper substitutions (3-86) becomes

$$j_L(x+y) = \sum_{L',l} i^{L'+l-L}(2L'+1)(LL'00|l0)^2 j_{L'}(x) j_l(y) \quad (3-87)$$

Letting,

$$x = 2kR_0 \sin\frac{\Theta}{2}$$

$$y = 2i\gamma \sin\frac{\Theta}{2}$$

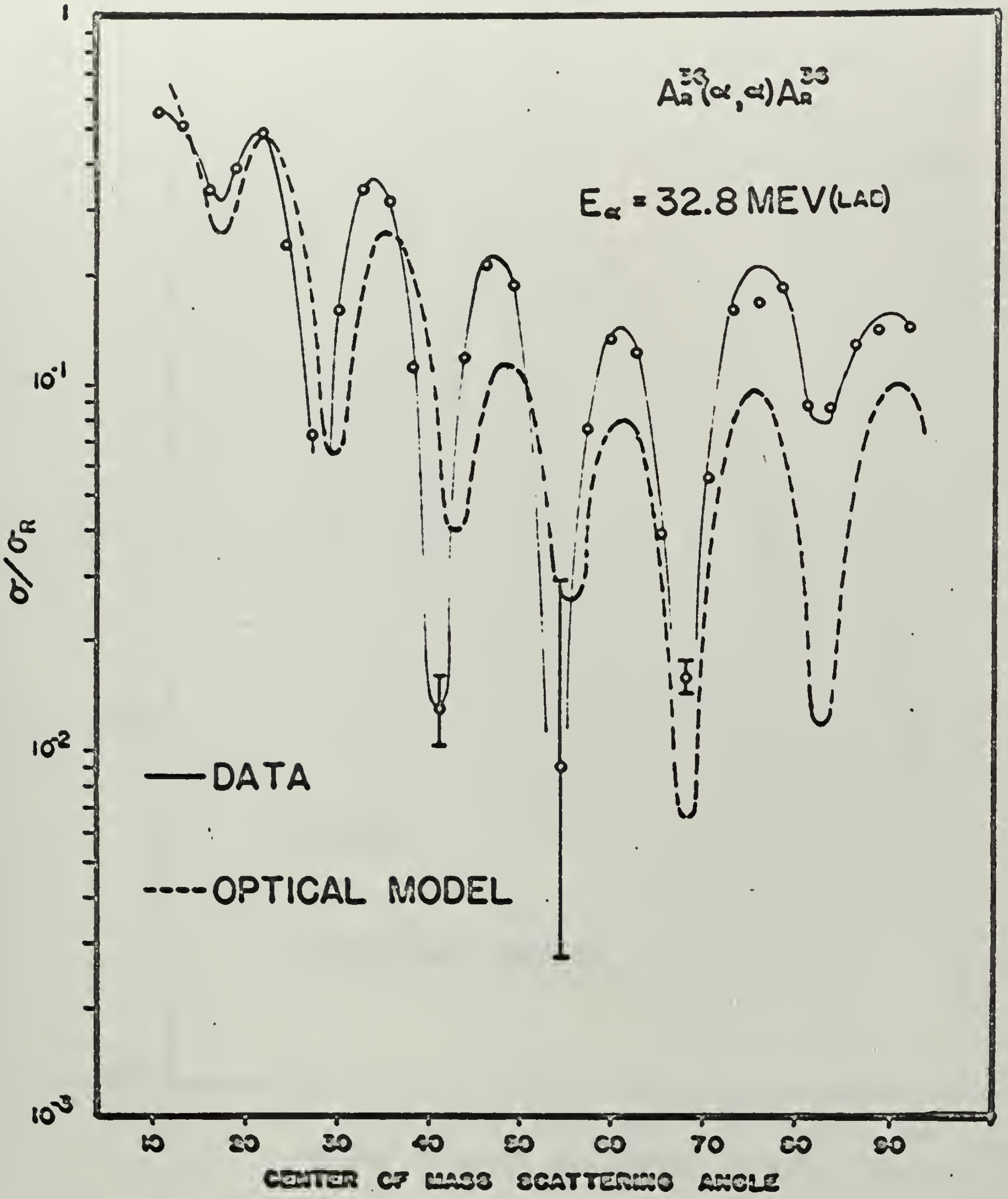
It is seen that·

$$j_L[2(kR_0 + i\gamma)\sin\frac{\Theta}{2}] = \sum_{L'l} i^{L'+L-l}(2L'+1) \quad (3-88)$$
$$\times (LL'00|l0)^2 j_l(2i\gamma \sin\frac{\Theta}{2}) j_{L'}(2kR_0 \sin\frac{\Theta}{2})$$

which is the desired result. The parameter γ evidently has the same effect on reducing the resolution of an angular momentum transfer measurement as the introduction of an angular anisotropy into the reaction.

Results

Optical model fits to the $Ar^{36}(\alpha,\alpha)Ar^{36}$ data are shown, for the lower energy case in figure (3-10), and in (3-11) for the higher energy case. These are included as the approximations in (3-68) are based on a study of the optical wave functions.and thus the ability of the optical model to predict the elastic



$Ar^{36}(\alpha,\alpha)Ar^{36}$
E_α = 32.8 MEV(LAB)
—DATA
----OPTICAL MODEL
σ/σ_R
1
10^{-1}
10^{-2}
10^{-3}
10
20
30
40
50
60
70
80
90
CENTER OF MASS SCATTERING ANGLE

FIGURE (3-10)

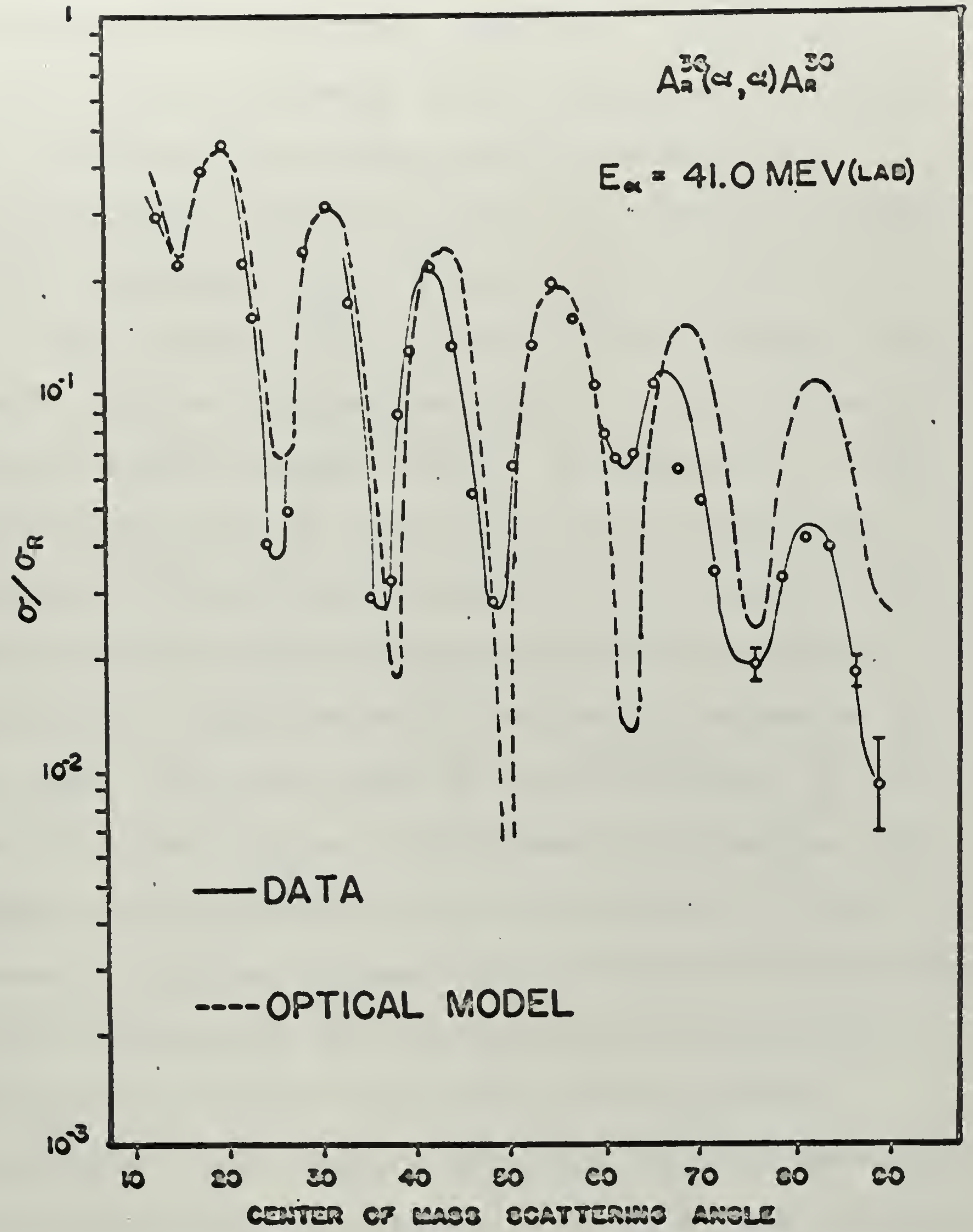

$A_R^{36}(\alpha,\alpha)A_R^{36}$
E_α = 41.0 MEV(LAB)
1
10^{-1}
10^{-2}
10^{-3}
σ/σ_R
—— DATA
---- OPTICAL MODEL
10
20
30
40
50
60
70
80
90
CENTER OF MASS SCATTERING ANGLE

FIGURE (3-11)

distributions should indicate the validity of these approximations for the data considered. The maxima and minima are reproduced very well, both in position as in the Blair formula, and in magnitude. Although not indicated, the optical model does well in predicting the behavior of the back angles. A radius of 5.6 Fermis was used in both cases.

Fits to the $Ar^{36}(\alpha, \alpha')Ar^{36*}$ data obtained using (3-81) and the appropriate choice of L are shown in figures (3-12) through (3-15). A single set of the parameters, λ and γ, were used to fit these data. They were $\lambda = 0.88$ and $\gamma = 0.90$ as was suggested by McCarthy and Pursey in the text of their article. Similarly a single value of the radius parameter R_0 was used. This was found to be 7.0 Fermis. It is seen that the angular predictions of the maxima and minima are comparable to the Blair theory. There is, however, improved agreement in the relative magnitudes of the theoretical and experimental distributions. The non-zero minima associated with the poorer resolution of the angular momentum transfer measurement is clearly shown. A systematic study of the parameters λ and γ was not made. The effect of γ, however, can be inferred from the previous study. Setting γ equal to zero removes the anisotropy and thus the minima will again reach zero. The interpretation

of the series in (3-88) was that the entire angular momentum transfer $L\hbar$ no longer went into the inelastic component of the scattering but only an amount $L'\hbar$. The rest was absorbed out of the beam or went into the elastic scattering component. Hence it is expected that γ would also affect the rate at which the cross section decreased with angle.

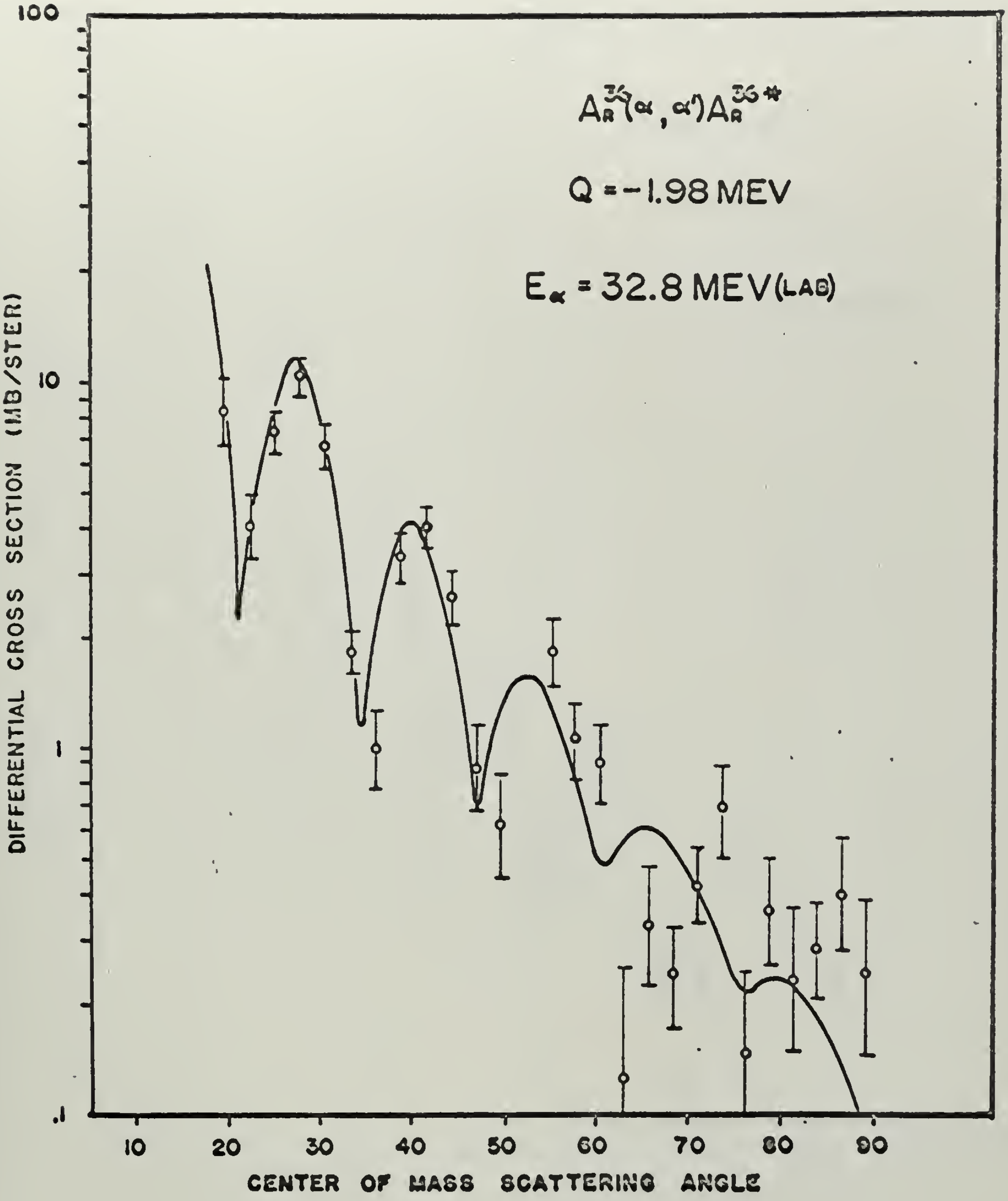
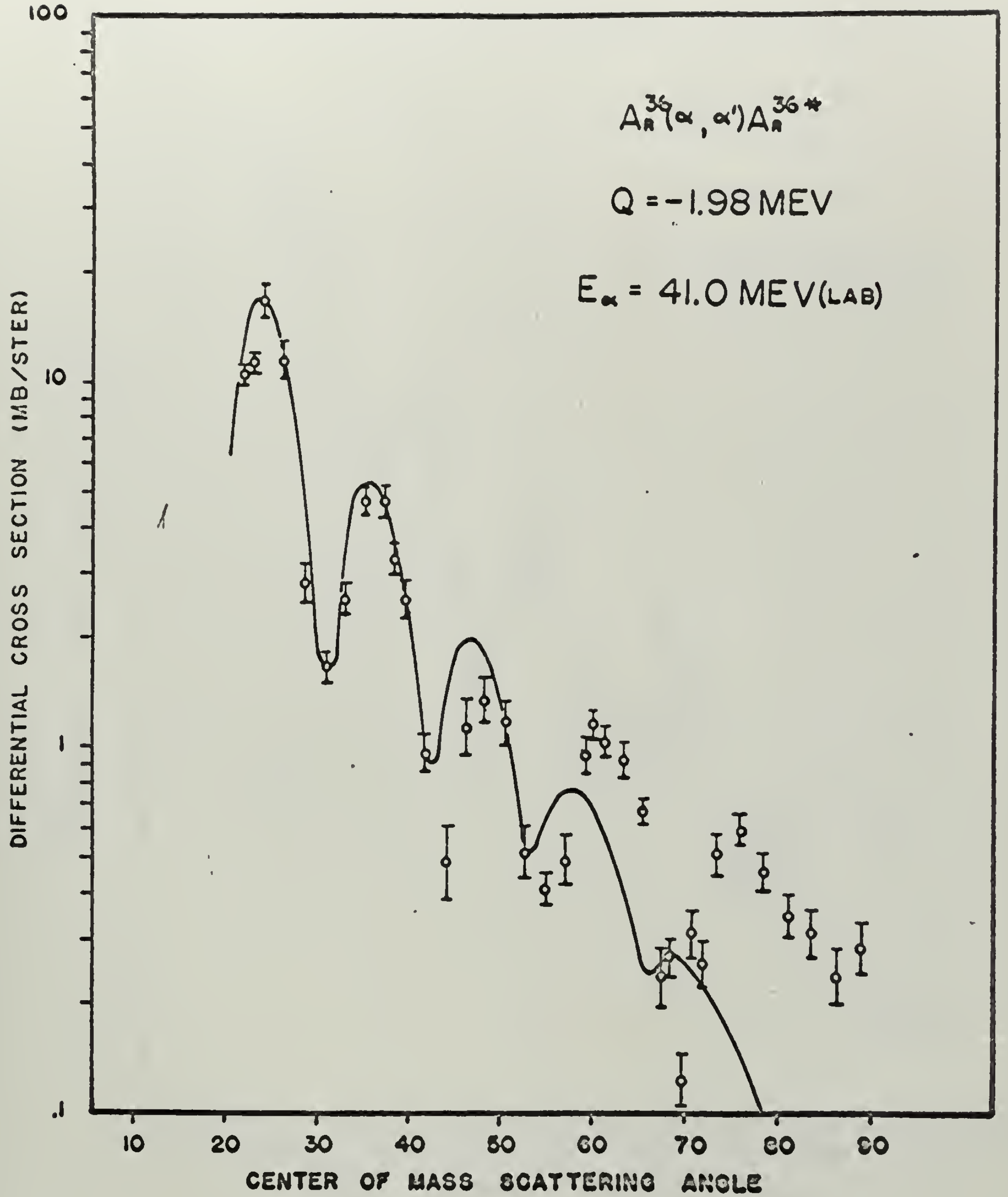

$Ar^{36}(\alpha, \alpha')Ar^{36*}$
Q = -1.98 MEV
E_α = 32.8 MEV(LAB)
DIFFERENTIAL CROSS SECTION (MB/STER)
100
10
1
.1
10
20
30
40
50
60
70
80
90
CENTER OF MASS SCATTERING ANGLE

FIGURE (3-12)



$A_R^{36}(\alpha,\alpha')A_R^{36*}$
Q = -1.98 MEV
E_α = 41.0 MEV(LAB)
DIFFERENTIAL CROSS SECTION (MB/STER)
100
10
1
.1
10
20
30
40
50
60
70
80
90
CENTER OF MASS SCATTERING ANGLE

FIGURE (3-13)

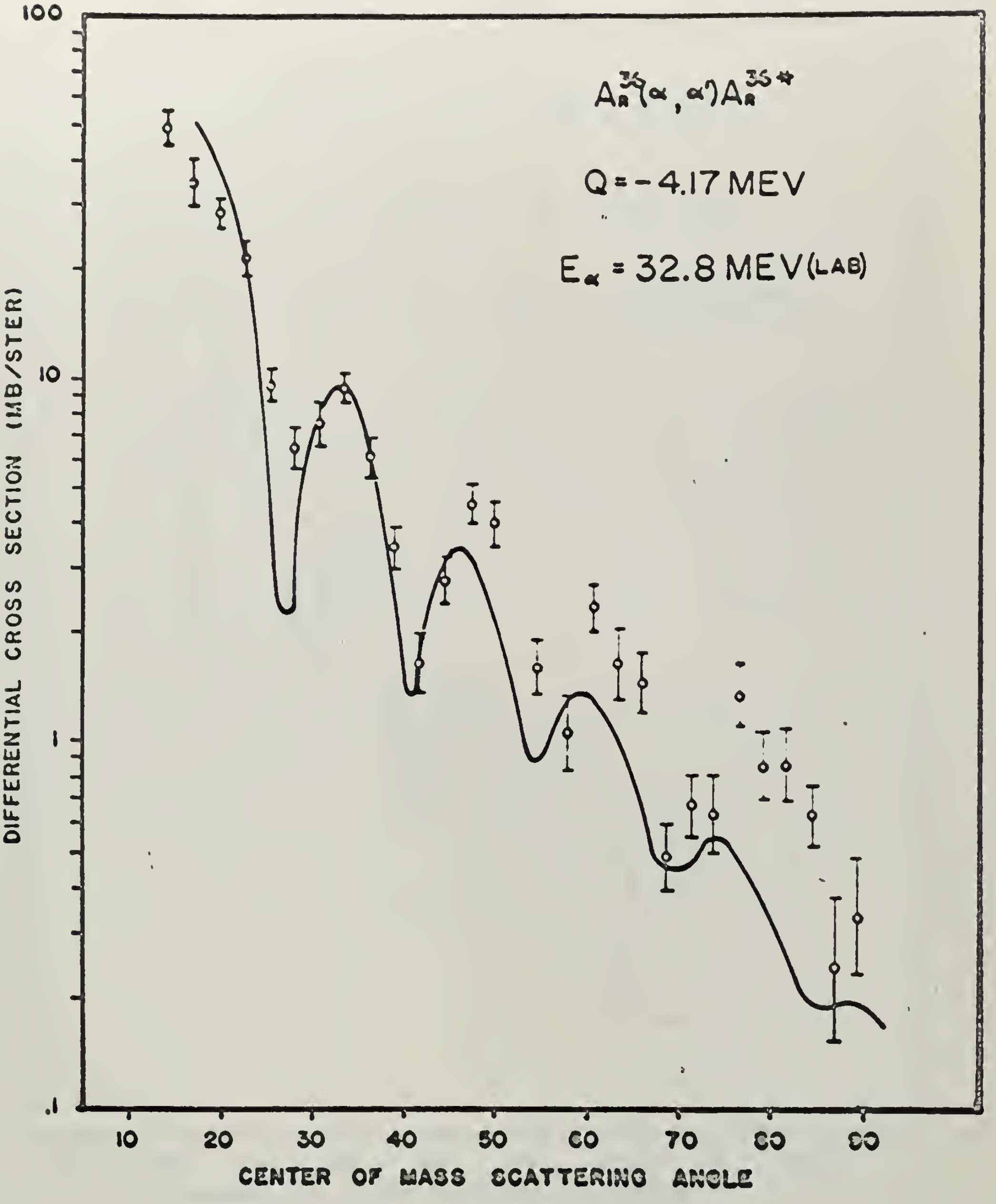

$A_R^{36}(\alpha,\alpha')A_R^{36*}$
Q = -4.17 MEV
E_α = 32.8 MEV (LAB)
DIFFERENTIAL CROSS SECTION (MB/STER)
100
10
1
.1
10
20
30
40
50
60
70
80
90
CENTER OF MASS SCATTERING ANGLE

FIGURE (3-14)

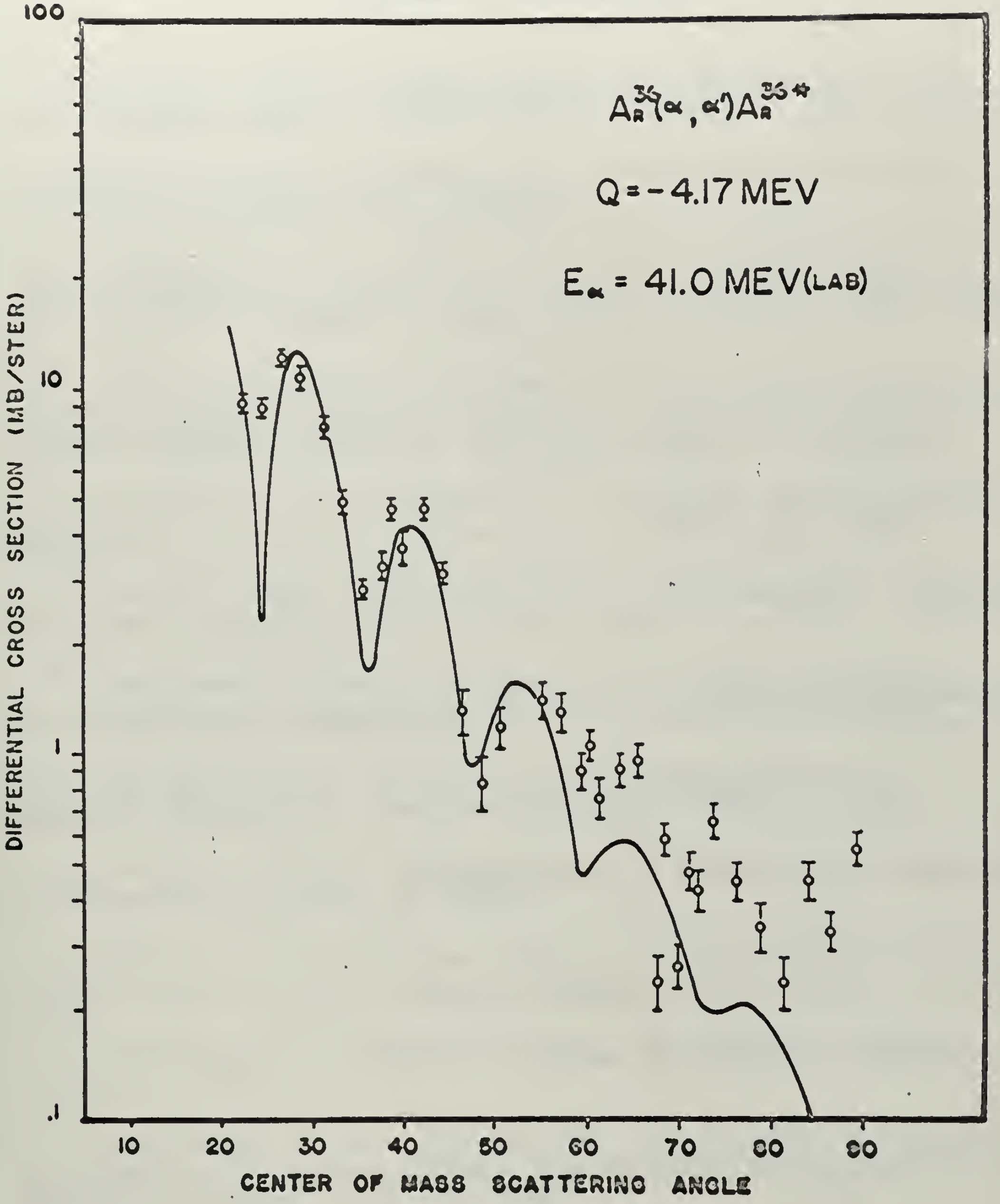

$A_R^{36}(\alpha,\alpha')A_R^{36*}$
Q = -4.17 MEV
E_α = 41.0 MEV(LAB)
100
10
1
.1
DIFFERENTIAL CROSS SECTION (MB/STER)
10
20
30
40
50
60
70
80
90
CENTER OF MASS SCATTERING ANGLE

FIGURE (3-15)

NOTES: CHAPTER III

1 Walsh, John B., Electromagnetic Theory and Engineering Applications, (Ronald Press, 1960), p. 36.

2 Blair, J. S., "Inelastic Diffraction Scattering," Physical Review, 115, 928(1959).

3 Drozdov, S. I., "The Scattering of fast Neutrons by non-spherical Nuclei," J. Exper. Theoret. Phys. USSR, 28, (1955), p. 734, p. 736..

4 Bohr, A., Mottelson, Ben R., "Collective and Individual-Particle Aspects of Nuclear Structure," Det Kgl. Danske Videnok. Selskab, Mat.-fsy. Medd., 27(1953).

5 Morse, P. M., Feshback, H., Methods of Theoretical Physics 2, (2 vols., McGraw-Hill, 1953), p. 1323.

6 Adler, Bohr, Huus, Mottelson, and Winther, Reviews of Modern Physics 28, 432(1956), Eq. II A. 25.

7 Margenau, Henery, Murphy, G. M., The Mathematics of Physics and Chemistry, (D. Van Nostrand Company, 1956).

8 Dicke, R. H., Wittke, J. P., Introduction to Quantum Mechanics, (Addison-Wesley, 1960), p. 61.

9 Becker, R. A., Introduction to Theoretical Mechanics, (McGraw-Hill, 1954), p. 240.

10 Endt, P. M., Van Der Leum, C., "Energy Levels of light Nuclei. III," Nuclear Physics, 34, (1962), p. 216.

11 Austern, N., Selected Topics in Nuclear Theory, (Vienna, 1963).

12 McCarthy, I. E., Pursey, D. L., "Simple Realistic Treatment of Nuclear Direct Interaction Processes," Physical Review, 122, 578(1961).

13 Preston, M. A., Physics of the Nucleus, (Addison-Wesley, 1962), Eq. A-73.

14 Ibid., Eq. A-31c

15 Ibid., Eq. A-32a

16 Rost, E., Austern, N., "Inelastic Diffraction Scattering," Physical Review, 120, 1375(1960).

17 Marr, G., Unpublished M.A. Thesis, Miami University, Oxford, Ohio, 1965.

18 Butler, S. T., "Direct Nuclear Reactions," Physical Review, 106, 272(1957).

19 Preston, Physics of the Nucleus, Eq. A-69.

20 Eisberg, R. M., McCarthy, I. E., Nuclear Physics, 10, 571(1959).

21 McCarthy, I. E., Nuclear Physics, 10, 583(1959), 574(1959).

22 McCarthy, Pursey, Physical Review, 122, (1961) p. 581.

APPENDIX A

THE CONFLUENT HYPERGEOMETRIC EQUATION

Kummer's equation is the hypergeometric equation often encountered by physicists in the solution of Schrodinger's equation.

It has the form

$$z\frac{d^2y}{dz^2} + (b-z)\frac{dy}{dz} - ay = 0 \qquad \text{(A-1)}$$

Solutions of this equation are known as confluent hypergeometric functions. This appendix will present only enough of the theory to provide an understanding of chapter II. The reader is referred to references [1] and [2] for a more complete development.

The Series Solution

It is assumed that equation (A-1) has a solution of the form

$$y = \sum_{n=0}^{\infty} c_n z^{k+n}$$

This is inserted into the differential equation resulting in a polynomial in z. For y to be a solution

for all values of z, the coefficients of all powers of z must be identically zero. With n set equal to zero, the coefficient of the lowest remaining power of z yields the indicial equation.

Solving this, gives

$$k = 0,\ 1-b \tag{A-2}$$

The first of these gives the desired solution which is regular at the origin. The following recurrence relationship for the coefficients is then obtained, assuming that c_o is non-zero.

$$C_{n+1} = \frac{(a+n)}{(n+1)(b+n)}\, C_n \tag{A-3}$$

With this, all the coefficients can be generated to within an arbitrary constant c_o and setting this equal to unity yields the following series

$$y = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\,\Gamma(b)\, z^n}{\Gamma(a)\,\Gamma(b+n)\, n!} \equiv {}_1F_1(a|b|z) \tag{A-4}$$

A Contour Integral Representation

The expression (A-4) is re-written

$$\begin{aligned} {}_1F_1(a|b|z) &= \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b-a)}{\Gamma(b+n)} \frac{z^n}{n!} \\ &= \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \sum_{n=0}^{\infty} B(a+n,\, b-a) \frac{z^n}{n!} \end{aligned} \tag{A-5}$$

where B(u,v) is the beta function and is defined[3]

$$B(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)} = \int_0^1 t^{u-1}(1-t)^{v-1}dt$$

A contour integral representation of the beta function can be shown to be

$$B(u,v) = (1-e^{2\pi i v})^{-1}\oint_{C_3} t^{u-1}(1-t)^{v-1}dt \tag{A-6}$$

where the contour C_3 is described in figure (A-1) and the phase defined, so that along the segment from 1 to 2,

$$\arg(t) = \arg(1-t) = 0$$

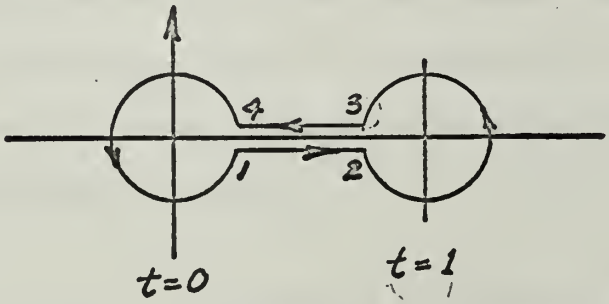


FIGURE (A-1)

For the expression to take on the same value after a complete circuit of the contour, it is seen that u + v must equal an integer. Equation (A-6) is inserted into (A-5) giving

$$_1F_1(a|b|z) = \frac{\Gamma(b)(1-e^{2\pi i(b-a)})^{-1}}{\Gamma(a)\Gamma(b-a)}\sum_{n=0}^{\infty}\frac{z^n}{n!}\oint_{C_3} t^{a+n-1}(1-t)^{b-a-1}dt$$

Interchanging the order of summation and integration, where it is seen that b must be an integer to satisfy the requirement that (A-6) be single valued and thus (A-7) be a solution to (A-1), this becomes

$$
\begin{aligned}
{}_1F_1(a|b|z) &= \frac{\Gamma(b)\left(1-e^{2\pi i(b-a)}\right)^{-1}}{\Gamma(a)\Gamma(b-a)} \oint_{C_3} \sum \frac{(zt)^n}{n!} t^{a-1}(1-t)^{b-a-1}\,dt \\
&= \frac{\Gamma(b)\left(1-e^{-2\pi i a}\right)^{-1}}{\Gamma(a)\Gamma(b-a)} \oint_{C_3} e^{zt}\, t^{a-1}(1-t)^{b-a-1}\,dt
\end{aligned}
\qquad \text{(A-7)}
$$

When the integrand of (A-7) is substituted back into the differential equation, the result can be expressed as a perfect differential and thus any contour which has the same value at its end points for which the integral converges leads to a solution. In particular, two of these will be considered yielding irregular solutions of (A-1).

Let $\quad {}_1F_1(a|b|z) = W_1(a|b|z) + W_2(a|b|z) \qquad \text{(A-8)}$

where

$$
W_{1,2}(a|b|z) = \frac{\Gamma(b)\left(1-e^{-2\pi i a}\right)^{-1}}{\Gamma(a)\Gamma(b-a)} \oint_{C_1, C_2} e^{zt}\, t^{a-1}(1-t)^{b-a-1}\,dt \qquad \text{(A-9)}
$$

The contours C_1 and C_2 are described in figure (A-2)

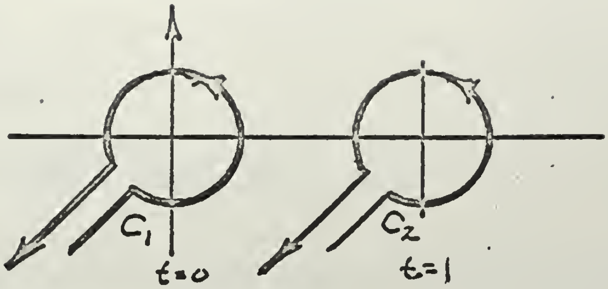


FIGURE (A-2)

where for convergence of the integrals,

$$\pi/2 < \arg z + \tau + 2n\pi < 3\pi/2$$

is the argument of the end points at infinity.

Consider the expression (A-9). By convention, the substitution zt = -u is made and

$$W_1(a|b|z) = \frac{\Gamma(b)(1-e^{-2\pi i a})(-z)^{-a}}{\Gamma(b)\Gamma(b-a)} \oint_{C_4} e^{-u} u^{a-1} \left(1+\frac{u}{z}\right)^{b-a-1} du \tag{A-10}$$

where the contour in the U-plane is shown in figure (A-3)

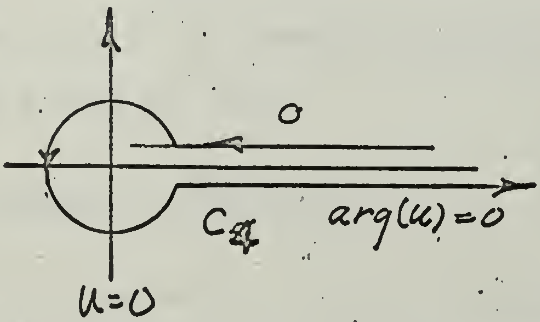


FIGURE (A-3)

Here, arg (u) $=$ 0 along the bottom portion of the curve along the real axis as was defined in figure (A-1). With this contour,

$$-\pi/2 < \arg z < \pi/2$$

Letting first t $=$ t $+$ 1 and then zt $=$ -u, an expression for $W_2(a|b|z)$ is obtained

$$W_2(a|b|z) = \frac{\Gamma(b)\,(e^{-2\pi i a}-1)^{-1} e^{z} z^{a-b}}{\Gamma(a)\,\Gamma(b-a)} \oint_{C_5} e^{-u}\left(1-\frac{u}{z}\right)^{a-1} u^{b-a-1}\, du \quad \text{(A-11)}$$

The contour C_5 is shown in figure (A-4)

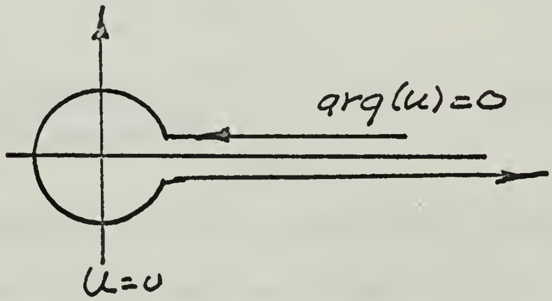


FIGURE (A-4)

where arg(u) $=$ 0 on the top portion of the curve along the real axis, again to be consistent with the phase convention in figure (A-1). To obtain convergence

$$-\pi/2 < \arg z < \pi/2$$

Asymptotic Expansion

Asymptotic series expressions will be obtained for the irregular solutions W_1 and W_2 and from these

and (A-8), an asymptotic expression for ${}_1F_1(a|b|z)$ will be found.

Consider expression (A-10) and its associated contour. The binomial expansion is used to obtain

$$\left(1+\frac{u}{z}\right)^{b-a-1} = \sum_{n=0}^{\infty} \frac{\Gamma(b-a)}{\Gamma(b-a-n)} \frac{\left(\frac{u}{z}\right)^n}{n!}$$

This is a series of negative powers of z and is thus an expansion for large z. This is inserted into (A-10) and the order of summation and integration reversed.

$$W_1(a|b|z) = \frac{\Gamma(b)(-z)^{-a}}{\Gamma(a)\,\Gamma(b-a)}\left(1-e^{-2\pi i a}\right)^{-1}$$
$$\times \sum_{n=0}^{\infty} \frac{\Gamma(b-a)\; z^{-n}}{\Gamma(b-a-n)\, n!} \oint_{C_4} e^{-u} u^{a+n-1}\, du \qquad \text{(A-11)}$$

The remaining integral and its associated contour C_4 given in figure (A-3) can be shown to be a contour integral representation of the gamma function.

$$\oint_{C_4} e^{-u} u^{a+n-1}\, du = \left(1-e^{-2\pi i a}\right)\Gamma(a+n)$$

Equation (A-11) becomes

$$W_1(a|b|z) = \frac{\Gamma(b)\,(-z)^{-a}}{\Gamma(b-a)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\,\Gamma(b-n)\, z^{-n}}{\Gamma(b-a-n)\,\Gamma(a)\, n!} \qquad \text{(A-12)}$$

This is put in a form where (-z) is the variable

of the series with the following identity which can be verified by induction.

$$\frac{\Gamma(-a)}{\Gamma(-a-n)} = (-1)^n \frac{\Gamma(a+n+1)}{\Gamma(a+1)} \tag{A-13}$$

where n is an integer. Equation (A-12) becomes

$$W_1(a|b|z) = \frac{\Gamma(b)\,(-z)^{-a}}{\Gamma(b-a)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\,\Gamma(a-b+n+1)\,(-z)^{-n}}{\Gamma(a)\,\Gamma(a-b+1)\,n!} \tag{A-14}$$

In expression (A-11), the binomial theorem is used to obtain

$$\left(1-\frac{u}{z}\right)^{a-1} = \sum_{n=0}^{\infty} \frac{\Gamma(a)\,\left(\frac{u}{-z}\right)^n}{\Gamma(a-n)\,n!}$$

Equation (A-11) becomes

$$W_2(a|b|z) = \frac{\Gamma(b)\,(e^{-2\pi i a}-1)^{-1}}{\Gamma(a)\,\Gamma(b-a)} e^{z} z^{a-b}$$

$$\times \sum_{n=0}^{\infty} \frac{\Gamma(b)\,(-z)^{-n}}{\Gamma(a-n)\,n!} \oint_{C_4} e^{-u} u^{b-a+n-1}\,du \tag{A-15}$$

As it was required that b be an integer, this can be written

$$W_2(a|b|z) \approx \frac{\Gamma(b)}{\Gamma(a)} e^{z} z^{a-b} \sum_{n=0}^{\infty} \frac{\Gamma(a)\,(-z)^{-n}}{\Gamma(a-n)\Gamma(b-a)n!}$$

$$\times\ (e^{2\pi i(b-a)} - 1)^{-1} \oint_{c_4} e^{-u} u^{b-a+n-1}\, du \tag{A-16}$$

The remaining integral and its related contour can be shown to be a contour integral representation of the gamma function

$$\oint_{c_4} e^{-u} u^{b-a+n-1} du = (e^{2\pi i(b-a)} - 1)\, \Gamma(b-a+n)$$

Equation (A-16) becomes

$$W_2(a|b|z) \approx \frac{\Gamma(b)}{\Gamma(a)} e^{z} z^{a-b} \sum \frac{\Gamma(a)\Gamma(b-a+n)\,(-z)^{-n}}{\Gamma(a-n)\,\Gamma(b-a)} \tag{A-17}$$

Letting $a \rightarrow -a$ in (A-13), equation (A-17) can be transformed in a manner similar to (A-14)

$$W_2(a|b|z) \approx \frac{\Gamma(b)}{\Gamma(a)} e^{z} z^{b-a} \sum_{n=0}^{\infty} \frac{\Gamma(n+1-a)\,\Gamma(n+b-a)\, z^{-n}}{\Gamma(1-a)\,\Gamma(b-a)\; n!} \tag{A-18}$$

For very large z, only the first term of the expansion is retained, thus equation (A-8) gives for the regular asymptotic solution

$$ {}_1F_1(a|b|z) \simeq \frac{\Gamma(a)}{\Gamma(b-a)}(-z)^{-a} + \frac{\Gamma(b)}{\Gamma(a)} e^{z} z^{a-b} \qquad \text{(A-19)} $$

where

$$ -\pi/2 < \arg z < \pi/2 $$

In the development used in chapter II, z takes on pure imaginary values, i.e.

$$ \arg(z) = \pm \pi/2 $$

In order to use expression (A-19) a limiting procedure will have to be conceived where arg(z) is allowed to approach $\pi/2$ or $-\pi/2$ in the direction of positively or negatively increasing angles, respectively. It will be assumed that this is done in all expressions therein.

NOTES: APPENDIX A

1 Slater, L. J., Confluent Hypergeometric Functions, (Cambridge University Press, 1960).

2 Whittaker, E. T., Watson, G. N., A Course in Modern Analysis, (Cambridge University Press, 1962).

3 Dwight, Herbert Bristol, Tables of Integrals and Other Mathematical Data 4th ed., (MacMillan Company, 1961), p. 211.

thesS925
Analysis of inelastic alpha particle sca
3 2768 002 06018 8
DUDLEY KNOX LIBRARY